

TEST #1 - SUMMER 2000TIME: 75 min.

Answer all 6 questions. Show all working and provide all reasoning. An unjustified answer will receive little credit.

- (20) 1. (a) Find the permutation which has  $\langle 3, 4, 2, 0, 1, 0 \rangle$  as its inversion sequence.  
(b) How many integers in the set  $\{1, 2, 3, \dots, 700\}$  are divisible by 10 or 12?
- (20) 2. (a) In how many ways can the letters of the word *SUCCESSES* be arranged?  
(b) Find the number of *integer solutions* of the equation  $x_1 + x_2 + x_3 = 10$  with  $x_1 \geq 2$ ,  $x_2 \geq 3$  and  $x_3 \geq -1$ .
- (20) 3. (a) Let  $A_1, A_2, \dots, A_n$  be n subsets of a universal set U. Define what are the *positive sets* and what are the *ultimate sets* with respect to this system of sets.  
(b) Use the binomial theorem to find  
$$2 \cdot \binom{n}{0} + 3 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 5 \cdot \binom{n}{3} + \dots + (n+2) \cdot \binom{n}{n}$$
- (20) 4. A sea captain has 8 identical gold coins, 10 identical silver coins, and 13 identical copper coins. In how many ways can he donate 25 coins to his church?  
[Leave your answer unsimplified]
- (20) 5. (a) A shop has huge numbers of Pepsi, Coke, Seven-up, and Canada-Dry sodas. In how many ways can a very thirsty student purchase 6 cans of sodas?  
(b) How many permutations of  $\{1, 2, 3, 4, 5, 6, 7\}$  have exactly 3 elements going to themselves.
- (20) 6. (a) Write down what the *Pigeon Hole Principle* says, in its *General Form*.  
(b) Prove that in any set of 6 students we can always find 3 mutual friends or 3 mutual strangers.

## SOLUTIONS TO TEST #1 - SUMMER 2000

$$1.(a) \langle i_1, \dots, i_6 \rangle = \langle 3, 4, 2, 0, 1, 0 \rangle$$

Step 1 :

$$\begin{matrix} " & 2 : & 6 \\ " & 3 : & 6, 5 \end{matrix}$$

$$\begin{matrix} " & 4 : & 4, 6, 3, 5 \end{matrix}$$

$$\begin{matrix} " & 8 : & 4, 6, 3, 5, 2 \end{matrix}$$

$$\begin{matrix} " & 6 : & 4, 6, 3, 1, 5, 2 \end{matrix}$$

So the permutation with inversion sequence

$$\langle 3, 4, 2, 0, 1, 0 \rangle \text{ is } \langle 4, 6, 3, 1, 5, 2 \rangle$$

(b) Let  $A$  = set of integers in  $\{1, \dots, 700\}$  div. by 10  
 and  $B$  = set of integers in  $\{1, \dots, 700\}$  div. by 12

$$\text{Then } |A| = \left\lfloor \frac{700}{10} \right\rfloor = 70$$

$$|B| = \left\lfloor \frac{700}{12} \right\rfloor = 58$$

$$\text{and } |A \cap B| = \left\lfloor \frac{700}{\text{lcm}(10, 12)} \right\rfloor = \left\lfloor \frac{700}{60} \right\rfloor = 11$$

Number of integers in  $\{1, \dots, 700\}$  divisible by 10

$$\text{or } 12 = |A \cup B|$$

$$= |A| + |B| - |A \cap B|$$

$$= 70 + 58 - 11$$

$$= 117$$

2.(a) Number of ways letters of SUCCESSES can be arranged = no. of perm. of [2.C, 2.E, 4.S, 1.U]

$$= \frac{9!}{2! 2! 4! 1!}$$

$$= \frac{9.8.7.6.5}{2.2.}$$

$$= 9.7.6.10 = 3780$$

(b) Number of integer solutions of  
 $x_1 + x_2 + x_3 = 10$  with  $x_1 \geq 2, x_2 \geq 3$  &  $x_3 \geq -1$

= Number of integer solutions of  
 $(y_1+2) + (y_2+3) + (y_3-1) = 10$  with  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

= Number of non-negative integer solutions of  
 $y_1 + y_2 + y_3 = 6$

$$= \binom{6+3-1}{3-1} = \binom{8}{2} = \frac{8.7}{2.1} = 28.$$

3. (a) A positive set with respect to  $A_1, \dots, A_n$  is any set of the form  $\cap_{i_1} A_{i_1} \cap_{i_2} \dots \cap_{i_k} A_{i_k}$  where  $\langle i_1, \dots, i_k \rangle$  is a subsequence of  $\langle 1, 2, \dots, n \rangle$ .

An ultimate set with respect to  $A_1, \dots, A_n$  is any set of the form  $X_1 \cap X_2 \cap \dots \cap X_n$  where  $X_i = A_i$  or  $A_i^c$ .

(b) From the Binomial theorem we know that

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x)^n.$$

$$\text{So } \binom{n}{0} \cdot x^2 + \binom{n}{1}x^3 + \binom{n}{2}x^4 + \dots + \binom{n}{n}x^{n+2} = x^2 \cdot (1+x)^n$$

Differentiating both sides w.r.t.  $x$  we get

$$\begin{aligned} 2 \cdot \binom{n}{0} \cdot x + 3 \cdot \binom{n}{1}x^2 + 4 \cdot \binom{n}{2}x^3 + \dots + (n+2) \cdot \binom{n}{n} \cdot x^{n+1} \\ = 2x \cdot (1+x)^n + x^2 \cdot n(1+x)^{n-1} \end{aligned}$$

Putting  $x=1$ , we get

$$\begin{aligned} 2 \binom{n}{0} + 3 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + \dots + (n+2) \cdot \binom{n}{n} \\ = 2 \cdot 1 \cdot 2^n + 1 \cdot n \cdot 2^{n-1} \\ = 2^{n-1} \cdot 4 + 2^{n-1} \cdot n \\ = 2^{n-1} \cdot (n+4) \end{aligned}$$

4. Number of ways he can donate 25 coins  
 = number of 25-comb. of  $[8.G, 10.S, 13.C]$

Let  $U$  = set of 25-comb. of  $M = [8.G, 10.S, 13.C]$

$A$  = set of 25-comb. of  $M$  with  $\geq 9G$

$B$  = " of  $M$  with  $\geq 11S$

$C$  = " of  $M$  with  $\geq 14C$ . Then

$A$  = set of 16-comb. of  $M$  plus 9.G

$B$  = " 14-comb of  $M$  plus 9.S

$C$  = " 11-comb of  $M$  plus 14.C

$A \cap B$  = set of 5-comb. of  $M$  plus 9.G & 11.S

$A \cap C$  = " 2-comb. of  $M$  plus 9.G & 14.C

$B \cap C$  = " 0-comb. of  $M$  plus 11.S & 14.C

$A \cap B \cap C = \emptyset$ . Hence

Number of 25-comb. of  $[8.G, 10.S, 13.C]$

$$= |(A^c \cap B^c \cap C^c)|$$

$$= |U| - |A| - |B| - |C| + |AB| + |AC| + |BC| - |ABC|$$

$$= \binom{25+3-1}{3-1} - \binom{25+3-1}{3-1} - \binom{14+3-1}{3-1} - \binom{11+3-1}{3-1}$$

$$+ \binom{5+3-1}{3-1} + \binom{2+3-1}{3-1} + \binom{0+3-1}{3-1} - 0$$

$$= \binom{27}{2} - \binom{18}{2} - \binom{16}{2} - \binom{13}{2} + \binom{7}{2} + \binom{4}{2} + \binom{2}{2}$$

5.(a) Let A, B, C, D represent Pepsi, Coke, Seven-Up and Canada-Dry sodas. Then

Number of ways student can purchase 6 sodas  
 = no. of 6-comb. of  $[\infty, A, \infty, B, \infty, C, \infty, D]$

$$= \binom{6+4-1}{4-1} = \binom{9}{3} = \frac{9!}{3!6!}$$

$$= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 84.$$

(b) First we must choose 3 elements which will go to themselves. There are  $\binom{7}{3}$  ways to do this. Then we have to derange the other 4 elements - there are  $D_4$  ways to do this. Hence

No. of perm. of  $\{1, 2, \dots, 7\}$  with exactly 3 elements going to themselves

$$= \binom{7}{3}, D_4$$

$$= \frac{7!}{3!4!} (4-1)(D_3 + D_2)$$

$$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}, 3, (2+1) = 7 \cdot 5 \cdot 9 = 315.$$

6. (a) Pigeon Hole Principle (General Form) : If  $k$  pigeons are placed in  $n$  holes then there will be a hole which contains at least  $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$  pigeons.

(b) Take one student, call her  $p_1$ . Let  
 $F$  = set of friends of  $p_1$  in the remaining 5  
 $S$  = set of strangers to  $p_1$  in the remaining 5.  
Then  $|S \cup F| = 5$ . So  $|S| \geq 3$  or  $|F| \geq 3$ .

Case(i) :  $|S| \geq 3$ . In this case either all the people in  $S$  are mutual friends or there are 2 people in  $S$  who are strangers. If all the people in  $S$  are friends we get 3 mutual friends. And if there were 2 people in  $S$  who are strangers we can add  $p_1$  to get 3 mutual strangers.

Case(ii)  $|F| \geq 3$ . In this case either all the people in  $F$  are mutual strangers or there are 2 friends in  $F$ . If all the people in  $F$  are mutual strangers in  $S$  we get 3 mutual strangers. And if there were 2 friends in  $F$  we can add  $p_1$  to get 3 mutual friends.

So in all cases we will be able to get 3 mutual friends or 3 mutual strangers in the group of 6 students.