

TEST #2 - SUMMER 2000TIME: 75 min.

Answer all 6 questions. A neat and clear presentation is essential for full credit. Show all working and provide all reasoning. An unjustified answer will receive little or no credit.

- (16) 1. Find the solution of the equation $a_{n+2} + 6a_{n+1} + 9a_n = 0$ with the initial conditions $a_0=1, a_1=3$.
- (24) 2. Find the general solution of the following difference equations
- $a_{n+2} - 2a_{n+1} + 5a_n = 8$
 - $a_{n+2} + 2a_{n+1} - 3a_n = 12$.
- (20) 3. (a) Let D_n be the no. of derangements of $\{1, 2, 3, \dots, n\}$ and let E_n be the no. of derangements of $\{1, 2, 3, \dots, n\}$ in which the first element is 2. Prove that $E_n = D_{n-1} + D_{n-2}$.
 (b) Hence prove that $D_n = (n-1)(D_{n-1} + D_{n-2})$.
- (20) 4. Use the method of generating functions to find the solution of the difference equation $a_n - a_{n-1} - 2a_{n-2} = 0$ with the initial conditions $a_0=1$ and $a_1=8$.
- (20) 5. (a) Starting with $(1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots$, find the generating function for $\langle 3n/2^n \rangle_{n \geq 0}$.
 (b) Let $S = [\infty, a, \infty, b, \infty, c]$ and h_n be the no. of n-combinations of S in which the no. of a's is odd, the no. of b's is a multiple of 3, and the no. of c's is even and ≥ 4 . Find the generating function of $\langle h_n \rangle_{n \geq 0}$.
- (Give your answers as functions, not as infinite series.)*
- (20) 6. (a) Define what are the standard generating function and the exponential generating function of a sequence $\langle h_n \rangle_{n \geq 0}$.
 (b) If $n \geq 2$, prove that in any group of n people we can always find 2 people who have the same number of friends in the group.

$$1. \quad a_{n+2} + 6a_{n+1} + 9a_n = 0$$

$$(E^2 + 6E + 9)a_n = 0$$

$$\text{Aux. eq. is } E^2 + 6E + 9 = 0$$

$$(E + 3)^2 = 0$$

$$\therefore E = -3 \text{ (twice)}$$

$$\text{So } a_n = (A + nB) \cdot (-3)^n$$

Since $a_0 = 1$ & $a_1 = 3$ we have

$$a_0 = 1 = (A + 0 \cdot B) \cdot (-3)^0$$

$$a_1 = 3 = (A + 1 \cdot B) \cdot (-3)^1$$

$$\therefore 1 = A$$

$$3 = (A + B) \cdot (-3) \Rightarrow A + B = -1$$

$$\therefore A = 1 \text{ & } B = -1 - 1 = -2$$

$$\text{So } a_n = (1 - 2n) \cdot (-3)^n$$

$$2. (a) \quad a_{n+2} - 2a_{n+1} + 5a_n = 0$$

$$(E^2 - 2E + 5)a_n = 0$$

$$E^2 - 2E + 5 = 0$$

$$\therefore E = \frac{-(-2) \pm \sqrt{4 - 4 \cdot 5}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

$$\therefore a_n^c = A \cdot (1+2i)^n + B \cdot (1-2i)^n$$

2(a) (cont.) Try $a_n^P = b$. Then
 $a_{n+1}^P = b$ and $a_{n+2}^P = b$

So $a_{n+2} - 2a_{n+1} + 5a_n = 8$ becomes

$$b - 2b + 5b = 8$$

$$\therefore 4b = 8 \Rightarrow b = 2$$

So $a_n^P = 2$.

$\therefore a_n = a_n^C + a_n^P = A \cdot (1+2i)^n + B \cdot (1-2i)^n + 2$
is the general solution.

(b) $a_{n+2} + 2a_{n+1} - 3a_n = 0$

$$E^2 + 2E - 3 = 0 \quad (\text{aux. eq.})$$

$$(E + 3)(E - 1) = 0 \Rightarrow E = -3 \text{ or } 1$$

$$\therefore a_n^C = A \cdot (1)^n + B \cdot (-3)^n = A + B \cdot (-3)^n$$

Since 1 is root of multiplicity 1 in the aux. eq.
we must try $a_n^P = b \cdot n^1 = b \cdot n$

$$a_{n+1}^P = b \cdot (n+1)$$

$$a_{n+2}^P = b \cdot (n+2)$$

So $a_{n+2} + 2a_{n+1} - 3a_n = 12$ becomes

$$b(n+2) + 2 \cdot b(n+1) - 3bn = 12$$

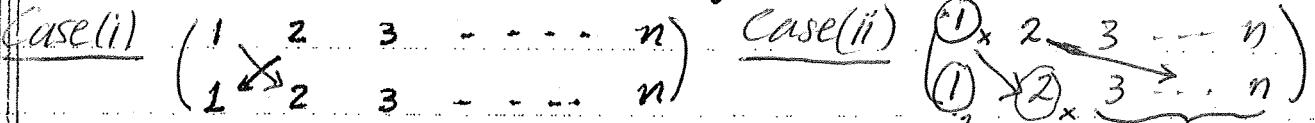
$$(b+2b-3b) \cdot n + 2b + 2b = 12$$

$$\therefore 4b = 12 \Rightarrow b = 3$$

$$\therefore a_n^P = 3n$$

So $a_n = a_n^C + a_n^P = A + B \cdot (-3)^n + 3n$.

3. (a) Consider an arbitrary derangement in E_n .



Either 2 goes to 1 or 2 goes to one of the elements $3, \dots, n$. In the first case 1 goes to 2 & 2 goes to 1 and the elements $3, \dots, n$ must be deranged. Since there are D_{n-2} ways of deranging $n-2$ elements this contributes D_{n-2} derangements to E_n . In the 2nd case 2 does not go to 1 - so if we pretend ^{the bottom} 1 is 2, this would be equivalent to deranging $2, 3, \dots, n$. In other words, the 2nd case will provide D_{n-1} derangements in E_n . Hence

$$E_n = D_{n-2} + D_{n-1} = D_{n-1} + D_{n-2}.$$

(b) Now in part (a) we considered that 2 went to 1. But we could have had any element $3, \dots, n$ going to 1. (We can't have 1 going to 1 because we want a derangement of $1, 2, \dots, n$). Since there were $n-1$ choices for the first element and each choice [such as the 2 in part (a)] produced $D_{n-1} + D_{n-2}$ derangements, the total number of derangements of $1, 2, \dots, n$ must be $(n-1) \cdot (D_{n-1} + D_{n-2})$. Hence

$$D_n = (n-1) \cdot (D_{n-1} + D_{n-2}).$$

4. Let $f(x)$ = the generating function of $\langle a_n \rangle_{n=0}^{\infty}$.
Then

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \\ -x f(x) &= -a_0 x - a_1 x^2 - \dots - a_{n-1} x^n - \dots \\ -2x^2 f(x) &= -2a_0 x^2 - \dots - 2a_{n-2} x^n - \dots \end{aligned}$$

Adding these equations we get

$$\begin{aligned} (1-x-2x^2) f(x) &= a_0 + (a_1-a_0)x + (a_2-a_1-2a_0)x^2 \\ &\quad + \dots + (a_n-a_{n-1}-2a_{n-2})x^n + \dots \\ &= a_0 + (a_1-a_0)x + 0x^2 + \dots + 0x^n + \dots \\ &= 1 + (8-1)x = 1+7x \end{aligned}$$

$$\therefore f(x) = \frac{1+7x}{1-x-2x^2} = \frac{1+7x}{(1+x)(1-2x)}$$

$$\text{Let } \frac{1+7x}{(1+x)(1-2x)} = \frac{A}{1+x} + \frac{B}{1-2x}$$

$$\text{Then } 1+7x = A(1-2x) + B(1+x)$$

Putting $x = -1$ gives us

$$1-7 = A[1-(-2)] + B \cdot 0 \Rightarrow -6 = 3A \Rightarrow A = -2$$

Putting $x = 1/2$ gives us

$$1+7/2 = A \cdot 0 + B \cdot \frac{3}{2} \Rightarrow \frac{9}{2} = \frac{3B}{2} \Rightarrow B = 3$$

$$\begin{aligned} \therefore f(x) &= \frac{-2}{1+x} + \frac{3}{1-2x} \\ &= (-2) \cdot [1 + (-x) + (-x)^2 + \dots + (-x)^n + \dots] \\ &\quad + 3 \cdot [1 + 2x + 2^2 x^2 + \dots + 2^n x^n + \dots] \end{aligned}$$

$$\begin{aligned} \therefore a_n &= \text{coefficient of } x^n \\ &= -2 \cdot (-1)^n + 3 \cdot 2^n. \end{aligned}$$

$$5(a) \quad 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}$$

Differentiating both sides we get

$$0 + 1 + 2x + \dots + nx^{n-1} + \dots = \frac{1}{(1-x)^2}$$

Multiplying both sides by $3x$ we get

$$0 + 3x + 3 \cdot 2x^2 + \dots + 3n \cdot x^n + \dots = \frac{3x}{(1-x)^2}$$

Replacing x by $x/2$ we get

$$0 + 3 \frac{x}{2} + 3 \cdot 2 \cdot \left(\frac{x}{2}\right)^2 + \dots + 3n \cdot \left(\frac{x}{2}\right)^n + \dots = \frac{3x/2}{(1-x/2)^2}$$

$$\therefore \sum_{n=0}^{\infty} 3n \cdot \frac{1}{2^n} \cdot x^n = \frac{3}{2} \cdot \frac{x}{(1-x/2)^2}$$

So gen. func. of $\left\langle \frac{3n}{2^n} \right\rangle_{n=0}^{\infty}$ is $\frac{3}{2} \frac{x}{(1-x/2)^n}$

(b) $h_n =$ coefficient of x^n in the expansion of

$$(x + x^3 + x^5 + \dots)(1 + x^3 + x^6 + \dots)(x^4 + x^6 + x^8 + \dots)$$

$$= x(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots) \cdot x^4(1 + x^2 + x^4 + \dots)$$

$$= x^5 \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4}$$

∴ generating function of $\langle h_n \rangle_{n=0}^{\infty}$ is

$$\frac{x^5}{(1-x^2)^2 \cdot (1-x^3)}$$

∴ 6. (a) The standard generating function of $\langle h_n \rangle_{n=0}^{\infty}$
is the function

$$f(x) = \sum_{n=0}^{\infty} h_n \cdot x^n$$

The exponential generating function of $\langle h_n \rangle_{n=0}^{\infty}$
is the function

$$g(x) = \sum_{n=0}^{\infty} h_n \cdot \frac{x^n}{n!}$$

(b) The proof splits into two cases.

Case (i) : There is a person in the group who
has no friends in the group.

In this case the maximum number of friends
a person can have is $n-2$. So the possibilities
are $0, 1, 2, \dots, n-2$ friends. Since there
are n people & $n-1$ possibilities, two people
must have the same number of friends by
the Pigeon Hole Principle

Case (ii) : There is no person in the group who
has no friends in the group.

In this case the possible number of friends
a person can have are $1, 2, 3, \dots, n-1$.
Since there are n people & $n-1$ possibilities
we must have two people, again, who have
the same number of friends by the
Pigeon Hole principle.