

TEST #1 - SUMMER 2002TIME: 75 min.

Answer all 6 questions. Show all working and provide all reasoning. An unjustified answer will receive little credit.
BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.

- (20) 1. (a) Find the permutation which has $\langle 5, 3, 1, 2, 1, 0 \rangle$ as its inversion sequence.
 (b) How many integers in the set $\{1, 2, 3, \dots, 2400\}$ are divisible by 12 or 20?
- (20) 2. (a) A girl lives 6 blocks west and 4 blocks south of her school. In how many ways can she walk the 10 blocks to school if her first block is always northwards?
 (b) Find the number of integer solutions of the equation $x_1 + x_2 + x_3 = 10$ with $x_1 \geq 2$, $x_2 \geq -5$ and $x_3 \geq 4$.
- (20) 3. (a) Write down what the *Inclusion-Exclusion Theorem* says.
 (b) A bank has huge numbers of quarters, dimes, nickels, and pennies only. In how many ways can a teller borrow a bag of 9 coins if she wants at least one of each kind ?
- (20) 4. (a) Use the *Binomial Theorem* to find the value of the sum

$$1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 3 \cdot \binom{n}{2} + \dots + (n+1) \cdot \binom{n}{n}$$

 (b) Find the coefficient of xy^3z^2 in the expansion of $(5x-y+2z)^6$.
- (20) 5. (a) Define what are the *ultimate sets* w.r.t. the subsets A_1, A_2, \dots, A_n of a universal set U .
 (b) How many permutations of $\{1, 2, 3, \dots, 6\}$ have no odd elements going to themselves.
 [You may leave your answer in terms of factorials and the binomial coefficients.]
- (20) 6. (a) Write down what the *Multinomial Theorem* says.
 (b) Let $S = \langle a_1, a_2, \dots, a_k \rangle$ be a sequence of k integers. Prove we can always find a section of S which adds up to a multiple of k .

1(a) Since there are 6 terms in the inversion sequence this is a permutation of $\{1, 2, 3, \dots, 6\}$

6

6 5

6 5 4

6 3 5 4

6 3 5 2 4

6 3 5 2 4 1

(b) Let $U = \{1, 2, 3, \dots, 2400\}$ and

$A = \text{set of numbers in } U \text{ divisible by 12}$

$B = \text{set in } U \text{ .. by 20}$

Then $A \cap B = \text{set of numbers in } U \text{ divisible by l.c.m}$

$$|A| = \left\lfloor \frac{2400}{12} \right\rfloor = 200, |B| = \left\lfloor \frac{2400}{20} \right\rfloor = 120 \quad \text{of } 12 \text{ & } 20.$$

$$\text{and } |A \cap B| = \left\lfloor \frac{2400}{60} \right\rfloor = 40.$$

So number of integers divisible by 12 or 20

$$= |A \cup B| = |A| + |B| - |A \cap B|$$

$$= 200 + 120 - 40 = 280$$

2(a) The girl has to walk 6 blocks east & 4 blocks north. Since she starts the first block northwards the number of ways she can walk the 10 blocks to school is the number of sequences of 6 E's and 4 N's with the first term being an N.

2(a) cont. $\langle N, -, -, -, \dots, - \rangle$
 1st 2nd 3rd 10th block

So answer = number of permutations of $[6E, 3N]$
 $= \frac{(6+3)!}{6! 3!} = \frac{9!}{6! 3!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84.$

(b) Let $x_1 = y_1 + 2$, $x_2 = y_2 - 5$ & $x_3 = y_3 + 4$. Then
 $x_1 + x_2 + x_3 = 10$ & $x_1 \geq 2$, $x_2 \geq -5$, $x_3 \geq 4$ (*)

becomes

$$(y_1 + 2) + (y_2 - 5) + (y_3 + 4) = 10 \text{ with } y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

$$\therefore y_1 + y_2 + y_3 = 10 + 5 - 4 - 2 = 9 \quad (**)$$

So number of solutions of (*)
 = number of solutions of (**)

$$= \binom{9+3-1}{3-1} = \binom{11}{2} = \frac{11!}{9!2!} = \frac{11 \cdot 10}{2 \cdot 1} = 55.$$

3(a) Let A_1, \dots, A_n be subsets of some finite universal set U . Then

$$|A_1^c \cap A_2^c \cap \dots \cap A_n^c| = \sum_{k=0}^n (-1)^k \left(\sum_{\substack{\text{pos. sets} \\ \text{of order } k}} |A_{i_1} \cap \dots \cap A_{i_k}| \right)$$

(b) Number of ways the teller can borrow 9 coins with the condition that she gets at least 1 of each kind
 = no. of 9-comb. of $[\infty \cdot q, \infty \cdot d, \infty \cdot n, \infty \cdot p]$
 with at least one q , one d , one n & one p
 = no. of 5-comb. of $[\infty \cdot q, \infty \cdot d, \infty \cdot n, \infty \cdot p]$
 (just add one of each kind to the 5-comb.)
 = $\binom{5+4-1}{4-1} = \binom{8}{3} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$

4(a) We know from the Binomial Theorem that

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \dots + \binom{n}{n} x^n y^0$$

Putting $y=1$, we get by multiplying both sides by x

$$\binom{n}{0} x + \binom{n}{1} x^2 + \binom{n}{2} x^3 + \dots + \binom{n}{n} x^{n+1} = x \cdot (x+1)^n$$

Differentiating both sides gives us

$$1 \cdot \binom{n}{0} + 2x \cdot \binom{n}{1} + 3x^2 \cdot \binom{n}{2} + \dots + (n+1)x^n \binom{n}{n} = 1 \cdot (x+1)^n \\ + x \cdot n \cdot (x+1)^{n-1}$$

Putting $x=1$ gives us

$$1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 3 \cdot \binom{n}{2} + \dots + (n+1) \cdot \binom{n}{n} = 1 \cdot 2^n + 1 \cdot n \cdot 2^{n-1} \\ = (n+2) \cdot 2^{n-1}$$

So our answer is $(n+2) \cdot 2^{n-1}$.

(b) In the expansion of $(5x-y+2z)^6$, the term involving xy^3z^2 will be

$$\binom{6}{1,3,2} (5x)^1 \cdot (-y)^3 \cdot (2z)^2$$

So the required coefficient will be

$$\frac{6!}{1!3!2!} (5) \cdot (-1)^3 \cdot 2^2 \\ = \frac{6 \cdot 5 \cdot 4^2}{2} (5) (-1) (4) = 30 \cdot (2) \cdot (-20) = -1200,$$

5(a) An ultimate set w.r.t to the subsets A_1, \dots, A_n is any set of the form $X_1 \cap X_2 \cap \dots \cap X_n$ where $X_i = A_i$ or A_i^c .

(b) Let $A_i = \text{set of all the permutations of } \{1, 2, \dots, 6\}$ in which $2i-1$ goes to itself. ($i=1, 2, 3$)

Also let U = set of all the permutations of $\{1, 2, \dots, 6\}$

5(b) Then $|U| = 6!$, $|A_1| = |A_2| = |A_3| = 5!$,
 $|A_1 A_2| = |A_1 A_3| = |A_2 A_3| = 4!$ and $|A_1 A_2 A_3| = 3!$
Now number of permutations in which no odd no. goes to itself
goes to itself $= |A_1^c \cap A_2^c \cap A_3^c|$
 $= |U| - |A_1| - |A_2| - |A_3| + |A_1 A_2| + |A_1 A_3| + |A_2 A_3| - |A_1 A_2 A_3|$
 $= 6! - \binom{3}{1} 5! + \binom{3}{1} 4! - 3!$

6.(a) The multinomial theorem says that

$$(x_1 + \dots + x_k)^n = \sum_{\{(n_1, \dots, n_k) : n_1 + \dots + n_k = n\}} \binom{n}{n_1, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

Here n_1, \dots, n_k are non-neg. integers and

$$\binom{n}{n_1, \dots, n_k} = n! / [(n_1!) (n_2!) \dots (n_k!)]$$

(b) Let $P = \{0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + \dots + a_k\}$

Then $|P| = k+1$. Now define $f: P \rightarrow \{0, \dots, k-1\}$
by $f(x) = x \pmod{k}$. Since $|P| = k+1$
we must have two elements with same modulus.

$$\therefore f(a_1 + \dots + a_j) \equiv f(a_1 + \dots + a_i) \pmod{k}$$

for some $0 \leq i < j \leq k$. Hence

$$(a_1 + a_2 + \dots + a_j) - (a_1 + \dots + a_i) \equiv 0 \pmod{k}$$

So

$$a_{i+1} + a_{i+2} + \dots + a_j \equiv 0 \pmod{k}$$

Thus $a_{i+1} + a_{i+2} + \dots + a_j$ is a multiple of k .

Hence we can always find a section of the sequence (a_1, \dots, a_k) which adds up to a multiple of k .