

TEST #2 - SUMMER 2002TIME: 75 min.

*Answer all 6 questions. An unjustified answer or failure to follow instructions will result in little credit. So show all working and provide all reasoning. BEGIN EACH QUESTION ON A SEPARATE PAGE.*

- (16) 1. Find the solution of the equation  $a_{n+2} + 6a_{n+1} + 9a_n = 0$  with the initial conditions  $a_0=4, a_1=3$ .
- (24) 2. Find the general solution of the following difference equations
- $a_{n+2} - 2a_{n+1} - 3a_n = 8$ .
  - $a_{n+1} - 3a_n = 12 \cdot 3^n$ .
- (20) 3. (a) Define the Stirling numbers of the First kind and the Stirling numbers of the Second kind.
- (b) Let  $h_n = 6n^2 - 4n + 3$ . So  $\langle h_n \rangle = \langle 3, 5, 19, 45, 83, 133, \dots \rangle$ . Find the zero diagonal of  $\langle h_n \rangle$  and a formula for the sum  $h_0 + h_1 + h_2 + \dots + h_n$ . (Simplify your answer)
- (20) 4. Use the method of generating functions to find the solution of the difference equation  $a_n + 3a_{n-1} + 8 = 0$  with the initial condition  $a_0=3$ .
- (20) 5. (a) Starting with  $(1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots$ , find the generating function for  $\langle (n+1)/(-2)^n \rangle_{n \geq 0}$ .
- (b) Let  $M = [2.a, 3.b, 3.c]$ . Use the method of exponential generating functions to find the number of 5-permutations of  $M$ . (Express your answers in terms of factorials and simplify as far as possible)
- (20) 6. (a) Define what is the zero diagonal (=zero column) of a sequence  $\langle h_n \rangle_{n \geq 0}$ .
- (b) Prove that in any group of 10 people we can always find 3 mutual friends or 4 mutual strangers. (You may use the fact that in any group of 6 people we can always find 3 mutual friends or 3 mutual strangers, if needed.)

$$1. \quad a_{n+2} + 6a_{n+1} + 9a_n = 0$$

$$(E^2 + 6E + 9)a_n = 0$$

$$\text{Aux. Eq. } E^2 + 6E + 9 = 0$$

$$(E+3)(E+3) = 0$$

$$\therefore E = -3 \text{ (twice)}$$

$$\therefore a_n = (A + Bn) \cdot (-3)^n$$

But  $a_0 = 4$  and  $a_1 = 3$ . So

$$4 = (A + B \cdot 0)(-3)^0 \Rightarrow A = 4$$

$$3 = (A + B)(-3)^1 \Rightarrow 3 = -3(A+B)$$

$$\Rightarrow 1 = -(4+B) \Rightarrow B = -1-4 = -5.$$

$$\therefore a_n = (4 - 5n) \cdot (-3)^n.$$

$$2(a) \quad a_{n+2} - 2a_{n+1} - 3a_n = 8 \quad (**)$$

$$\text{Homog. Eq. } a_{n+2} - 2a_{n+1} - 3a_n = 0 \quad (*)$$

$$(E^2 - 2E - 3) = 0$$

$$(E+1)(E-3) = 0$$

$$\therefore (a_n)_c = A \cdot (-1)^n + B \cdot (3)^n$$

Try  $(a_n)_p = b$ . Then  $(a_{n+1})_p = b$  &  $(a_{n+2})_p = b$

$$\text{So } (*) \text{ becomes } b - 2b - 3b = 8$$

$$\therefore -4b = 8 \quad \text{so } b = -2$$

$$\therefore a_n = (a_n)_c + (a_n)_p = A(-1)^n + B \cdot (3)^n - 2$$

$$2(b) \quad a_{n+1} - 3a_n = 12 \cdot (3)^n \quad (**)$$

$$\text{Homog. Eq.} \quad a_{n+1} - 3a_n = 0$$

$$\text{Aux. Eq.} \quad E - 3 = 0 \quad \Rightarrow E = 3$$

$$\therefore (a_n)_c = A \cdot (3)^n.$$

Since 3 is a root of the aux. eq., try

$$(a_n)_p = b \cdot n \cdot (3)^n.$$

$$(a_{n+1})_p = b \cdot (n+1) \cdot 3^{n+1} = (3b + 3bn) \cdot 3^n.$$

So (\*\*) becomes

$$(3bn + 3b) \cdot 3^n - 3 \cdot (bn \cdot 3^n) = 12 \cdot 3^n$$

$$\therefore 3b \cdot 3^n = 12 \cdot 3^n$$

$$\therefore b = 4.$$

$$\text{So } a_n = (a_n)_c + (a_n)_p = A \cdot 3^n + 4n \cdot 3^n$$

3. (a) The stirling numbers of the 1st kind are the unique integers  $\begin{Bmatrix} p \\ k \end{Bmatrix}$  such that

$$[n]_p = \sum_{k=0}^p (-1)^{p-k} \begin{Bmatrix} p \\ k \end{Bmatrix} \cdot n^k$$

The stirling numbers of the 2nd kind are the unique integers  $\{p\}_k$  such that

$$n^p = \sum_{k=0}^p \{p\}_k [n]_k$$

$$(b) \quad h_n = 6n^2 - 4n + 3$$

$n$	0	1	2	3	4	5
$\Delta^0 h_n$	3	5	19	45	83	133
$\Delta^1 h_n$	2	14	26	38	50	.
$\Delta^2 h_n$	12	12	12	12	.	.
$\Delta^3 h_n$	0	0	0	-	.	.

3(b) So zero column =  $\langle 3, 2, 12, 0, \dots \rangle$

$$\therefore h_k = 3\binom{k}{0} + 2\binom{k}{1} + 12\binom{k}{2}$$

$$\begin{aligned} \text{So } \sum_{k=0}^n h_k &= 3\binom{n+1}{0+1} + 2\binom{n+1}{1+1} + 12\binom{n+1}{2+1} \\ &= 3\frac{(n+1)}{1!} + 2\frac{(n+1)(n)}{2!} + 12\frac{(n+1)(n)(n-1)}{3!} \\ &= 3(n+1) + (n+1)n + 2(n+1)(n^2-n) \\ &= (n+1)(3+n+2n^2-2n) \\ &= (n+1)(2n^2-n+3). \end{aligned}$$

4. Let  $f(x)$  = generating function of  $\langle a_n \rangle_{n=0}^\infty$ . Then

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \\ 3x f(x) &= 3a_0 x + 3a_1 x^2 + \dots + 3a_{n-1} x^n + \dots \\ \frac{8}{1-x} &= 8 + 8x + 8x^2 + \dots + 8x^n + \dots \end{aligned}$$

$$\begin{aligned} \therefore (1+3x)f(x) + \frac{8}{1-x} &= (a_0 + 8) + (a_1 + 3a_0 + 8)x + (a_2 + 3a_1 + 8)x^2 \\ &\quad + \dots + (a_n + 3a_{n-1} + 8)x^n + \dots \\ &= (3+8) + 0 \cdot x + 0 \cdot x^2 + \dots + 0 \cdot x^n + \dots \end{aligned}$$

$$\therefore (1+3x)f(x) = 11 - \frac{8}{1-x} = \frac{3-11x}{1-x}$$

$$\therefore f(x) = \frac{3-11x}{(1-x)(1+3x)} = \frac{A}{1+3x} + \frac{B}{1-x}$$

$$\therefore 3-11x = A(1-x) + B(1+3x)$$

Putting  $x=-1/3$  gives us  $3 + \frac{11}{3} = A \cdot \frac{4}{3}$

$$\therefore \frac{20}{3} = \frac{4A}{3} \Rightarrow A = 5.$$

4. Putting  $x = 1$  gives us  $3 - 11 = B \cdot (1+3)$

$$\therefore -8 = 4B \Rightarrow B = -2$$

$$\begin{aligned} \therefore f(x) &= \frac{5}{1+3x} + \frac{-2}{1-x} = \frac{5}{1-(-3x)} + \frac{-2}{1-x} \\ &= 5 \cdot [1 + (-3x) + (-3x)^2 + \dots + (-3x)^n + \dots] \\ &\quad - 2 [1 + x + x^2 + \dots + x^n + \dots] \end{aligned}$$

$$\begin{aligned} \therefore a_n &= \text{coeff of } x^n \text{ in the expansion of } f(x) \\ &= 5 \cdot (-3)^n - 2. \end{aligned}$$

5.(a)  $1 + x + x^2 + \dots + x^n + \dots = (1-x)^{-1}$

Differentiating both sides we get

$$0 + 1 + 2x + 3x^2 + \dots + nx^{n-1} + (n+1)x^n + \dots = (-1)^2 \cdot (1-x)^{-2}$$

$$\therefore 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots = (1-x)^{-2}$$

Replacing  $x$  by  $-\frac{x}{2}$  we get

$$1 + 2\left(-\frac{x}{2}\right) + 3\left(-\frac{x}{2}\right)^2 + \dots + \frac{n+1}{(-2)^n} \cdot x^n + \dots = \left(1 + \frac{x}{2}\right)^{-2}$$

$$= \frac{4}{(2+x)^2}$$

$\therefore$  gen. function of  $\left(\frac{n+1}{(-2)^n}\right)_{n=0}^{\infty}$  is  $\frac{4}{(2+x)^2}$ .

(b) Number of 5-permutations of  $[2.a, 3.b, 3.c] = M$  is the coefficient of  $\frac{x^5}{5!}$  in the factorial expansion of

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right) \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right).$$

$$1 \cdot \frac{x^2}{2!} \cdot \frac{x^3}{3!} + 1 \cdot \frac{x^3}{3!} \cdot \frac{x^2}{2!} + \frac{x}{1!} \cdot \frac{x}{1!} \cdot \frac{x^3}{3!} + \frac{x}{1!} \cdot \frac{x^2}{2!} \cdot \frac{x^2}{2!} +$$

$$\frac{x}{1!} \cdot \frac{x^3}{3!} \cdot \frac{x}{1!} + \frac{x^2}{2!} \cdot \frac{x}{1!} \cdot \frac{x^2}{2!} + \frac{x^2}{2!} \cdot 1 \cdot \frac{x^3}{3!} + \frac{x^2}{2!} \cdot \frac{x^3}{3!} \cdot 1 + \frac{x^2}{2!} \cdot \frac{x^2}{2!} \cdot \frac{x}{1!}$$

$$\text{Ans} = \left( \frac{4}{2!3!} + \frac{2}{3!} + \frac{3}{2!2!} \right) \cdot 5!$$

6 (a) The zero column of  $\langle h_n \rangle_{n=0}^{\infty}$  is defined to be the sequence  $\langle \Delta^k h_0 \rangle_{k=0}^{\infty} = \langle \Delta^0 h_0, \Delta^1 h_0, \Delta^2 h_0, \dots, \Delta^k h_0, \dots \rangle$

(b) Choose one of the 10 people & call her  $p_1$ . Let  $F = \text{friends of } p_1$  &  $S = \text{strangers to } p_1$ .

Then  $|F \cup S| = 9$  and  $F \cap S = \emptyset$ . So either  $|F| \geq 4$  or  $|S| \geq 6$ .

Case(i) :  $|F| \geq 4$ . In this case either there are 2 mutual friends in  $F$  or everyone in  $F$  are mutual strangers. If we have 2 friends in  $F$  we add  $p_1$  to get 3 mutual friends. And if everyone in  $F$  are mutual strangers we get 4 mutual strangers because  $|F| \geq 4$ .

Case(ii) :  $|S| \geq 6$  : In this case we know that there are 3 mut. friends or 3 mutual strangers in  $S$  by a theorem in class. If there are 3 mut. friends then we got our 3 mut. friends. And if there are 3 mut. strangers in  $S$ , we can add  $p_1$  to get 4 mut. strangers. So in both cases we get 3 mut. friends or 4 mut. strangers.

5. (b) Alternative solution.

Ans = coeff. of  $\frac{x^5}{5!}$  in the expansion of

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right) \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

$$= \left(1 + \frac{x}{1} + \frac{x^2}{2}\right) \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \left(\frac{1}{4} + \frac{2}{6}\right)x^4 + \frac{1}{12}x^5 + \dots\right)$$

$$= 1 + \dots + \left[\frac{1}{2}\left(\frac{4}{3}\right) + \frac{1}{4}\left(\frac{3}{12} + \frac{4}{12}\right) + \frac{1}{12}(1)\right]x^5 + \dots$$

$$= 1 + \dots + \left(\frac{17}{12}, 5!\right) \frac{x^5}{5!} + \dots$$

$$\text{So Ans} = \frac{17}{12} \cdot 5! = \frac{17}{12} \cdot (120) = 170.$$