

Answer all 6 questions. An unjustified answer will receive little credit.

(15) 1. (a) Define what is the inversion sequence of a permutation of $\{1, \dots, n\}$.

(b) Find the permutation with inversion sequence $\langle 4, 1, 3, 3, 0, 1, 0 \rangle$.

(15) 2. (i) In how many ways can the letters of "LETTERS" be arranged?

(ii) How many of the permutations of $\{1, 2, 3, \dots, 7\}$ have exactly 4 elements going to themselves.

(15) 3. Find with justification

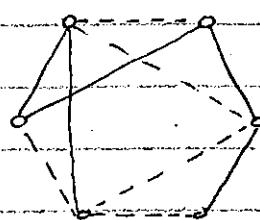
$$(a) \binom{n}{0} = 2^0 \binom{n}{1} + 2^1 \binom{n}{2} - 2^3 \binom{n}{3} + \dots + (-1)^n 2^n \binom{n}{n}$$

$$(b) \binom{n}{0} = \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} - \frac{1}{4} \binom{n}{3} + \dots + (-1)^n \frac{1}{n+1} \binom{n}{n}$$

(15) 4. How many of the integers between 1 and 1,000 (inclusive) are divisible by 8, 10, or 15?

(20) 5. A young lady has 4 quarters, 6 nickels, and 5 dimes. In how many ways can she give her little sister 13 coins. (LEAVE YOUR ANSWER UNSIMPLIFIED)

(20) 6. The 15 possible lines between the pairs of vertices of a regular hexagon are coloured in red or blue, randomly. Prove that the figure always contain a triangle with all three edges red or with all three edges blue.



10 possible lines shown here

1. (a) The inversion sequence of a permutation of $\{1, 2, \dots, n\}$ is the sequence $\langle i_1, \dots, i_n \rangle$ defined by $i_k = \text{no. of terms bigger than } k \text{ that precede } k \text{ in the permutation.}$ ($k = 1, 2, \dots, n$)

$$(b) \langle i_1, \dots, i_k \rangle = \langle 4, 1, 3, 3, 0, 1, 0 \rangle$$

Step 1. 7

" 2. 7 6

" 3. 5 7 6

" 4. 5 7 6 4

" 5. 5 7 6 3 4

" 6. 5 2 7 6 3 4

" 7. 5 2 7 6 1 3 4 = permutation.

2. (i) Here we want the number of permutations of the multi-set $[1.L, 2.E, 2.T, 1.R, 1.S]$. This is

$$\frac{7!}{1! 2! 2! 1! 1!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 2} = 1260.$$

(ii) First we choose 4 elements of $\{1, 2, \dots, 7\}$ which will go to themselves and then we derange the remaining 3 elements. Now there are $\binom{7}{4}$ ways of choosing the 4 elements and D_3 ways of deranging the remaining 3. So our answer is:

$$\binom{7}{4} \cdot D_3 = \frac{7!}{4! 3!} 3! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \right]$$

$$= \frac{7 \cdot 6 \cdot 5}{3!} \cdot 2 = 70.$$

3. From the binomial theorem we know that

$$\binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \dots + \binom{n}{n} x^n = (x+y)^n \quad (*)$$

(a) If we put $y=1$ and $x=-2$ in $(*)$ we obtain

$$\binom{n}{0} + \binom{n}{1}(-2) + \binom{n}{2}(-2)^2 + \binom{n}{3}(-2)^3 + \dots + \binom{n}{n}(-2)^n = (-2+1)^n$$

$$\therefore \binom{n}{0} - 2\binom{n}{1} + 2^2\binom{n}{2} - 2^3\binom{n}{3} + \dots + (-1)^n 2^n \binom{n}{n} = (-1)^n.$$

(b) If we put $y=1$ in $(*)$ we get

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n = (x+1)^n$$

Integrating both sides from -1 to 0 gives

$$\left[\binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \binom{n}{3}\frac{x^4}{4} + \dots + \binom{n}{n}\frac{x^{n+1}}{n+1} \right]_0^{-1} = \left[\frac{(x+1)^{n+1}}{n+1} \right]_0^{-1}$$

$$[0] - \left[-\binom{n}{0} + \frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} - \dots + \frac{(-1)^{n+1}}{n+1}\binom{n}{n} \right] = \left[\frac{1}{n+1} \right] - [0]$$

$$\therefore \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \dots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}.$$

4 Let A, B , and C be the set of numbers in $\{1, 2, 3, \dots, 1000\}$ which are divisible by 8, 10, and 15 respectively. Then
Then $A \cap B =$ numbers divisible by l.c.m. of 8 and 10, and
so on. We want $|A \cup B \cup C|$. This is given by

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \left\lfloor \frac{1000}{8} \right\rfloor + \left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{40} \right\rfloor - \left\lfloor \frac{1000}{120} \right\rfloor - \left\lfloor \frac{1000}{30} \right\rfloor + \left\lfloor \frac{1000}{120} \right\rfloor$$

$$= \frac{100}{8} + 100 + 66 - 25 - 8 - 33 + 8$$

$$= 266 - 33 = 233.$$

Here $\lfloor x \rfloor =$ the largest integer lesser than or equal to x .

5. Here we want the number of 13-combinations
of the multi-set $M = [4.q, 5.d, 6.n]$.

Let $S = [\infty.q, \infty.d, \infty.n]$ and put

U = set of all 13-comb. of S .

$A = \text{ " " } \text{ with } \geq 5q's.$

$B = \text{ " " } \text{ " } \geq 6d's.$

$C = \text{ " " } \text{ " } \geq 7n's.$

Then $A = \text{ set of all 8-comb. of } S + 5q's$

$B = \text{ " } 7\text{-comb. " } + 6d's$

$C = \text{ " } 6\text{-comb. " } + 7n's$

$A \cap B = \text{ " } 2\text{-comb. " } + 5q's \& 6d's$

$A \cap C = \text{ " } 1\text{-comb. " } + 5q's \& 7n's$

$B \cap C = \text{ " } 0\text{-comb. " } + 6d's \& 7n's$

$A \cap B \cap C = \emptyset$

So the number of 13-comb. of M is

$$|A^c \cap B^c \cap C^c| = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| \\ + |B \cap C| - |A \cap B \cap C|$$

$$= \binom{13+3-1}{13} - \binom{8+3-1}{8} - \binom{7+3-1}{7} - \binom{6+3-1}{6}$$

$$+ \binom{2+3-1}{2} + \binom{1+3-1}{1} + \binom{0+3-1}{0} - 0.$$

$$= \binom{15}{13} - \binom{10}{8} - \binom{9}{7} - \binom{8}{6} + \binom{4}{2} + \binom{3}{1} + \binom{2}{0}.$$

6. Choose one of the six vertices of the hexagon and call it p_1 . Then the other 5 vertices can be divided into 2 subgroups:

B = set of vertices joined to p_1 with a blue line.

R = " " red "

Now

we must have $|B| \geq 3$ or $|R| \geq 3$ b.c. $|BUR|=5$.

Case (i) : $|B| \geq 3$

In this case either there are two points in B that are joined by a blue line or all the points in B are joined by red lines. If two points of B are joined by a blue line we just add p_1 to get a blue triangle. If all the points in B are joined by red lines we get a red triangle from any 3 of the points in B .

Case (ii) : $|R| \geq 3$

In this case either there are two points in R joined by a blue line or all the points in R are joined by blue lines. If two points of R are joined by a red line we just add p_1 to get a red triangle. If all the points in R are joined by blue lines we get a blue triangle from any 3 of the points in R .

Thus in all the cases we can find a red triangle or a blue triangle.