

Answer all 6 questions. An unjustified answer will receive little credit.

- (15) 1. Starting with  $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$ , find generating functions for the sequences  $\langle (-2)^n \rangle_{n=0}^{\infty}$  and  $\langle 2n \rangle_{n=0}^{\infty}$ .

- (20) 2. Find the general solution of the following difference equations

$$(a) \quad x_{n+2} - 2x_{n+1} + 4x_n = 0$$

$$(b) \quad 4x_{n+2} + 4x_{n+1} + x_n = 3n$$

- (20) 3. Find the solution of the equation

$$x_{n+2} + x_{n+1} - 6x_n = 4$$

with the initial conditions  $x_0 = 3, x_1 = 2$ .

- (15) 4. By using generating functions, find the solution of the equation  $a_n - 2a_{n-1} + 1 = 0$  with initial condition  $a_0 = 0$ .

- (15) 5. Let  $G$  be a connected planar graph with  $n$  vertices and  $e$  edges which divides the plane into  $f$  regions. If  $G$  has no cycles of length 3, 4 or 5 show that

$$(a) \quad 3f \leq e \quad \text{and} \quad (b) \quad 2e \leq 3n - 6$$

- (15) 6. Let  $G$  be a graph with  $n$  vertices. Prove that  $G$  must have two vertices with the same degree.

SOLUTIONS TO TEST #2 - SUMMER 1990

1. We know that  $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \quad (*)$

So

$$\begin{aligned}\frac{1}{1-(-2x)} &= 1 + (-2x) + (-2x)^2 + \dots + (-2x)^n + \dots \\ &= 1 - 2x + 2^2 x^2 + \dots + (-2)^n x^n + \dots\end{aligned}$$

$\therefore$  the generating function of  $\langle (-2)^n \rangle_{n=0}^{\infty}$  is

$$\frac{1}{1-(-2x)} = \frac{1}{1+2x}.$$

Also by differentiating both sides of  $(*)$  we obtain

$$\frac{-1}{(1-x)^2} \cdot (-1) = 0 + 1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots$$

$$\therefore \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \dots + nx^{n-1} + \dots$$

Multiplying both sides by  $2x$  gives us

$$\frac{2x}{(1-x)^2} = 2.0 + 2.1.x + 2.2.x^2 + 2.3.x^3 + \dots + 2n.x^n + \dots$$

$\therefore$  the generating function of  $\langle 2n \rangle_{n=0}^{\infty}$  is  $\frac{2x}{(1-x)^2}$ .

2. (a)  $x_{n+2} - 2x_{n+1} + 4x_n = 0$

Aux. eq<sup>n</sup>:  $E^2 - 2E + 4 = 0$

$$\therefore E = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}i$$

So the general solution is

$$x_n = A \cdot (1+\sqrt{3}i)^n + B \cdot (1-\sqrt{3}i)^n$$

2. (b)

$$4x_{n+2} + 4x_{n+1} + x_n = 3n \quad \dots (*)$$

Homog. eq<sup>n</sup> is :  $4x_{n+2} + 4x_{n+1} + x_n = 0$

Aux. eq<sup>n</sup> is :  $4E^2 + 4E + 1 = 0$

$$\therefore (2E+1)(2E+1) = 0. \text{ So } E = -\frac{1}{2} \text{ (twice)}$$

The complementary solution is thus

$$x_n^c = (A + Bn) \cdot \left(\frac{1}{2}\right)^n$$

Now suppose that a particular solution is of the form  $x_n^p = an + b$ . Then

$$x_{n+1}^p = a(n+1) + b$$

$$x_{n+2}^p = a(n+2) + b$$

So (\*) becomes

$$4[a(n+2) + b] + 4[a(n+1) + b] + an + b = 3n$$

$$\therefore 9an + (12a + 9b) = 3n + 0$$

$$\text{So } 9a = 3 \quad (\text{coeff. of } n)$$

$$\text{and } 12a + 9b = 0 \quad (\text{const. term})$$

$$\therefore a = \frac{1}{3} \text{ and } b = -12a = -12 \cdot \frac{1}{3} = -4$$

A quick check shows that  $\frac{1}{3}n - \frac{4}{9}$  is indeed a solution of (\*).

So the general solution of (\*) is

$$x_n = x_n^c + x_n^p = (A + Bn) \cdot \left(\frac{1}{2}\right)^n + \frac{1}{3}n - \frac{4}{9}$$

3. We have  $x_{n+2} + x_{n+1} - 6x_n = 4 \quad \dots (*)$

Homog. eq<sup>n</sup> is :  $x_{n+2} + x_{n+1} - 6x_n = 0$ .

Aux. eq<sup>n</sup> is :  $E^2 + E - 6 = 0$

$$\therefore (E+3)(E-2) = 0 \quad \therefore E = 2 \text{ or } -3$$

$$\therefore x_n^c = A \cdot 2^n + B \cdot (-3)^n$$

3. Suppose that a particular solution is of the form  $x_n^P = c$ . Then  $x_{n+1}^P = c$ ,  $x_{n+2}^P = c$ . So (\*) becomes

$$c + c - 6c = 4$$

$$\therefore -4c = 4 \text{ and so } c = -1.$$

A quick check shows that  $x_n = -1$  is indeed a solution of (\*). Thus the general solution is

$$x_n = x_n^C + x_n^P = A \cdot 2^n + B \cdot (-3)^n - 1.$$

But  $x_0 = 3$  and  $x_1 = 2$ , so

$$A + B - 1 = 3 \quad (1)$$

$$2A - 3B - 1 = 2 \quad (2)$$

$$(1) \times 3: \quad 5A + 3B - 3 = 9 \quad (3)$$

$$(2) + (3): \Rightarrow 5A - 4 = 11$$

$$\therefore A = 3 \text{ and } B = 3 - A + 1 = 3 - 3 + 1 = 1.$$

So the solution is  $x_n = 3 \cdot 2^n + (-3)^n - 1$ .

4. Let  $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

$$\text{Then } -2x f(x) = -2a_0 x - 2a_1 x^2 - \dots - 2a_{n-1} x^n - \dots$$

$$\text{Also } \frac{1}{1-x} = +1 + 1 \cdot x + 1 \cdot x^2 + \dots + 1 \cdot x^n + \dots$$

$$\therefore (1-2x) f(x) + \frac{1}{1-x} = a_0 + 1 + (a_1 - 2a_0 + 1) \cdot x + (a_2 - 2a_1 + 1) x^2 + \dots + (a_n - 2a_{n-1} + 1) x^n + \dots$$

$$= a_0 + 1, \text{ since } a_n - 2a_{n-1} + 1 = 0, n \geq 1$$

$$= +1, \text{ since } a_0 = 0.$$

$$\therefore (1-2x) f(x) = +1 + \frac{-1}{1-x} = \frac{+1(1-x) - 1}{1-x} = \frac{-x}{1-x}$$

$$4. \quad \therefore f(x) = \frac{-x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$$

$$\text{So } -x = A(1-x) + B(1-2x)$$

$$\text{Putting } x = \frac{1}{2} \text{ gives } -\frac{1}{2} = A(1-\frac{1}{2}) + B \cdot 0$$

$$\therefore -\frac{1}{2} = \frac{1}{2}A \quad \therefore A = -1$$

$$\text{Putting } x = 1 \text{ gives } -1 = A \cdot 0 + B(1-2)$$

$$\therefore -1 = -B \quad \therefore B = 1.$$

So

$$f(x) = \frac{-1}{1-2x} + \frac{1}{1-x}$$

$$= -(1 + 2x + (2x)^2 + \dots + (2x)^n + \dots) \\ + (1 + x + x^2 + \dots + x^n + \dots)$$

So

$$a_n = \text{coefficient of } x^n \text{ in the expansion of } f(x) \\ = -2^n + 1$$

5. Since  $G$  is a connected planar graph, we know that  $f = e+2-n$  (Euler's formula)

- (a) Since  $G$  has no cycles of length 3, 4 or 5 we see that, in any planar representation of  $G$ , each face is bounded by at least 6 edges.  
 $\therefore (\text{No. of faces}) \times 6 \leq \text{No. of edges counted from faces}$   
 $\therefore 6f \leq 2e$  because each edge is counted in at most 2 faces.  
 $\therefore 3f \leq e.$

- (b) Substituting for  $f$  by using Euler's formula gives:  $3(e+2-n) \leq e$ .  
 $\therefore 3e - e \leq 3n - 6$ . Thus  $2e \leq 3n - 6$ .

6. There are two cases:

Case (i) :  $G$  has no vertex of degree 0.

In this case the maximum possible degree is  $n-1$  (because a vertex can only be adjacent to the other  $n-1$  vertices at best). So the set of possible degrees is

$$\{1, 2, 3, \dots, n-1\}$$

Since we have  $n$  vertices, it follows from the Pigeon Hole Principle that two vertices must have the same degree.

Case (ii) :  $G$  has a vertex of degree 0.

In this case a vertex can only be adjacent to at most  $n-2$  other vertices (because it and the vertex of degree 0 are excluded). So the maximum possible degree is  $n-2$ , and hence the set of possible degrees is

$$\{0, 1, 2, \dots, n-2\}$$

We again have  $n-1$  choices and  $n$  vertices.

By the pigeon Hole Principle it follows that two vertices must have the same degree.