

# Ch 1 - Combinations & Permutations of Sets

①

80. What is Combinatorics?

Combinatorics is the study of the properties of Discrete Structures. These structures are usually finite - but they can also be infinite. So in a certain sense Combinatorics includes Graph Theory, Number Theory, Optimization Theory, & even Abstract Algebra. In this course, MAD 4203 we will avoid, as far as possible, topics in these other fields - for the simple reason that they will be covered in those courses. There are 5 basic kinds of problems that are usually asked in Combinatorics.

1. Existence Problems: In these kinds of problems one is being asked if an object, with a prescribed property, exists.

Ex. 1 Is there a positive integer  $n$  such that the sum of all the divisors of  $n$  equals  
(i)  $n$ , (ii)  $2n$ , (iii)  $3n$ ?

(i) It is easy to see that  $n=1$  is the only number with the sum of its divisors equal to  $n$ , because every integer  $k > 1$  has at least two divisors, 1 &  $k$ , and these two add up to more than  $k$  already.

(ii) 28 is an example of a positive integer  $n$

with the sum of its divisors equal to  $2n$  because <sup>(2)</sup> the divisors of 28 are 1, 2, 4, 7, 14 & 28 and  $1+2+4+7+14+28 = 56 = 2(28)$ .

(iii) We will leave the problem of determining whether or not an integer  $n$ , with the sum of its divisors equal to  $2n$ , exists as a question for our kind reader to investigate.

2. Construction problems: In these kinds of problems, one is being asked to construct 1 or more objects with a prescribed property in a systematic way. Naturally these kinds of problems are related to Existence problems - because once we construct one object we will have shown that such kinds of objects exist. Sometimes we also have a proof by contradiction that such objects exist but we need to find an actual object. A construction will produce one or more such objects.

Ex.2 Find a systematic way of finding positive integers  $n$  with the sum of the divisors of  $n$  equal to  $2n$ . Such integers are called perfect.

Sol. We can easily verify that if  $2^p - 1$  is prime, then  $2^{p-1}(2^p - 1)$  is perfect because

$$T + 2^0(2^p - 1) + 2^1(2^p - 1) + \dots + 2^{p-1}(2^p - 1) = 2^{p-1}(2^p - 1) \cdot 2$$

So we have a systematic way of finding

Here  $T = 2^0 + 2^1 + 2^2 + \dots + 2^{p-1} = 2^p - 1$ .

lots of perfect numbers. let's give some.

$2^2 - 1 = 3$  is prime, so  $2^{2-1} \cdot (2^2 - 1) = 6$  is perfect.

$2^3 - 1 = 7$  is prime, so  $2^{3-1} \cdot (2^3 - 1) = 28$  is perfect.

$2^5 - 1 = 31$  is prime, so  $2^{5-1} \cdot (2^5 - 1) = 496$  is perfect.

$2^7 - 1 = 127$  is prime, so  $2^{7-1} \cdot (2^7 - 1) = 64 \cdot 127$  is perfect.

$2^n - 1$  is not prime, so we can't say that

$2^{n-1} \cdot (2^n - 1)$  is perfect. By the way we only looked at  $2^p - 1$  when  $p$  is prime because if  $k$  is not prime, then  $2^k - 1$  is not prime.

3. Optimization problems. In these kinds of problems one is asked to find all "best" (in a certain sense) objects which have a prescribed property. Usually there is only one "best" object with a given property but sometimes there can be more than one "best" object.

Ex 3 Find the best approximation to  $\pi$  which is of the form  $p/q$  with  $p, q \in \mathbb{Z}^+$  and  $p, q$  having at most 3 digits.

Sol. We know that there are several approximations to  $\pi$ .

$$\frac{31}{10} = 3.1, \frac{63}{20} = 3.15, \frac{157}{50} = 3.14, \dots, \frac{355}{113} = 3.1415929$$

Now  $\pi = 3.14159265\dots$  and from the Theory of continued fractions, we can say that  $355/113$  will be the best approximation. ( $355/113$  agrees with  $\pi$  to 6 decimal places.)

4. Counting Problems: In these kinds of problems, <sup>(4)</sup> one is being asked to count the number of objects with a particular property.

Ex.4 How many numbers in the set  $\{1, 2, 3, \dots, n\}$  are divisible by 6?

Sol. The answer is  $\lfloor n/6 \rfloor$  because the subset of all the numbers in  $\{1, 2, 3, \dots, n\}$  that are divisible by 6 is  $\{6(1), 6(2), 6(3), \dots, 6(k)\}$  where  $k = \lfloor n/6 \rfloor$ .  
Here  $\lfloor n/6 \rfloor =$  largest integer  $\leq n/6$ .

5. Listing (or Enumeration) Problems: In these kinds of problems, one is being asked to list (or enumerate) all the objects with a particular property. Naturally listing problems are closely related to counting problems — because if we can list all the objects with a given property, then we can easily count the number of such objects.

Ex.5 List all the numbers in the set  $\{1, 2, 3, \dots, n\}$  that are divisible by 12 and are also perfect squares.

Sol. The set of all such numbers is  
 $\{[6(1)]^2, [6(2)]^2, [6(3)]^2, \dots, [6(k)]^2\}$   
where  $k = \lfloor \sqrt{n}/6 \rfloor = \lfloor \sqrt{n/36} \rfloor$ .

Note: Most of the problems in Combinatorics are of type 4 or type 5 — so Combinatorics is called the "Art of Counting."

## §1. Basic counting techniques:

There are three basic principles which we use in counting. The first one is the Equivalence principle.

Notation: If  $A$  is a finite set, then we shall use the notation  $|A|$  to denote the size of  $A$  (no. of elements of  $A$ ). So "1.1" is a function from the finite sets to  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Note that  $|\emptyset| = 0$ . Recall also that a function  $f: A \rightarrow B$  is a bijection if  $f$  is an injection (i.e., one-to-one) and if  $f$  is also a surjection (i.e., onto).

The Equivalence Principle: If we can find a bijection from  $A$  to  $B$ , then  $|A| = |B|$ . In particular if we can find a bijection from  $A$  to  $\{1, 2, 3, \dots, n\}$  then  $|A| = n$ , because by definition  $|\{1, 2, 3, \dots, n\}| = n$ .

- Ex. 1 How many elements of  $U = \{1, 2, 3, \dots, 1000\}$  are
- divisible by 12?
  - perfect squares?
  - divisible by both 10 and 12?
  - perfect squares which are divisible by 12?

Sol. Let  $A = \{x \in U : x \text{ is divisible by } 12\}$   
 $B = \{x \in U : x \text{ is a perfect square}\}$   
 and  $C = \{x \in U : x \text{ is divisible by } 10\}$ .

(a) Then  $A = \{12(1), 12(2), 12(3), \dots, 12(\lfloor \frac{1000}{12} \rfloor)\}$  (6)  
 So number of elements of  $U$  that are divisible by 12  
 $= |A| = \lfloor \frac{1000}{12} \rfloor = 83$  by the equivalence principle  
 because  $f: A \rightarrow \{1, 2, 3, \dots, 83\}$ ,  $f(k) = k/12$  is  
 a bijection.

(b) Also  $B = \{(1)^2, (2)^2, (3)^2, \dots, (\lfloor \sqrt{1000} \rfloor)^2\}$ .  
 So no. of elements of  $U$  that are perfect squares  
 $= |B| = \lfloor \sqrt{1000} \rfloor = 31$  by the equivalence principle  
 bec.  $g: B \rightarrow \{1, 2, 3, \dots, 31\}$ ,  $g(k) = \sqrt{k}$  is a bijection.

(c) No. of elements of  $U$  that are divisible by both 10 & 12  
 $= |A \cap C|$ . Now  $A \cap C = \{x \in U: x \text{ is divisible by}$   
 the l.c.m.  $(12, 10)\} = \{x \in U: x \text{ is divisible by } 60\}$   
 $= \{60(1), 60(2), \dots, 60(\lfloor \frac{1000}{60} \rfloor)\}$   
 So our answer will be  
 $|A \cap C| = \lfloor \frac{1000}{60} \rfloor = \lfloor \frac{100}{6} \rfloor = 16$ .

(d) Our answer will be  $|A \cap B|$ . Now  
 $A \cap B = \{x \in U: x \text{ is divisible by } 12 \text{ \& } x \text{ is a perfect sq.}\}$   
 $= \{x \in U: x \text{ is a perfect sq. \& } x = (2^2 \cdot 3) \cdot k \text{ with } k \in \mathbb{Z}^+\}$   
 $= \{x \in U: x = 2^2 \cdot 3^2 \cdot l^2 \text{ for some } l \in \mathbb{Z}^+\}$   
 $= \{(6l)^2: (6l)^2 \leq 1000 \text{ \& } l \in \mathbb{Z}^+\}$   
 $= \{[6(1)]^2, [6(2)]^2, [6(3)]^2, \dots, [6(\lfloor \frac{\sqrt{1000}}{6} \rfloor)]^2\}$   
 So answer  $= |A \cap B| = \lfloor \frac{\sqrt{1000}}{6} \rfloor = \lfloor \frac{\sqrt{1000}}{6} \rfloor = 5$ .

Next, we have our second fundamental counting principle, the Addition Principle.

The Addition Principle: If a set  $A$  can be partitioned into  $k$  disjoint non-empty subsets  $A_1, \dots, A_k$  then  $|A| = |A_1| + |A_2| + \dots + |A_k|$ . In particular, if all the  $A_i$ 's are all of the same size, then  $|A| = k \cdot |A_1|$ . (7)

Ex. 2 How many elements of  $U = \{1, 2, 3, \dots, 1000\}$  are

- (a) not divisible by 12?
- (b) divisible by 12 or 10?
- (c) divisible by 12 or are perfect squares?
- (d) divisible by neither 12 nor 10?

Sol. (a) Our answer is  $|A^c|$ . Now  $U = A \cup (A^c)$  is a partition of  $U$  into disjoint subsets. So  $|U| = |A| + |A^c|$ .  $\therefore |A^c| = |U| - |A| = 1000 - 83 = 917$ . So our answer will be 917.

(b) Our answer is  $|A \cup C|$ . Now  $A \cup C = (A - A \cap C) \cup (A \cap C) \cup (C - (A \cap C))$ .  
 $\therefore |A \cup C| = [|A| - |A \cap C|] + |A \cap C| + [|C| - |A \cap C|]$   
 $= |A| + |C| - |A \cap C|$   
 $= \lfloor \frac{1000}{12} \rfloor + \lfloor \frac{1000}{10} \rfloor - \lfloor \frac{1000}{60} \rfloor = 83 + 100 - 16 = 167$ .

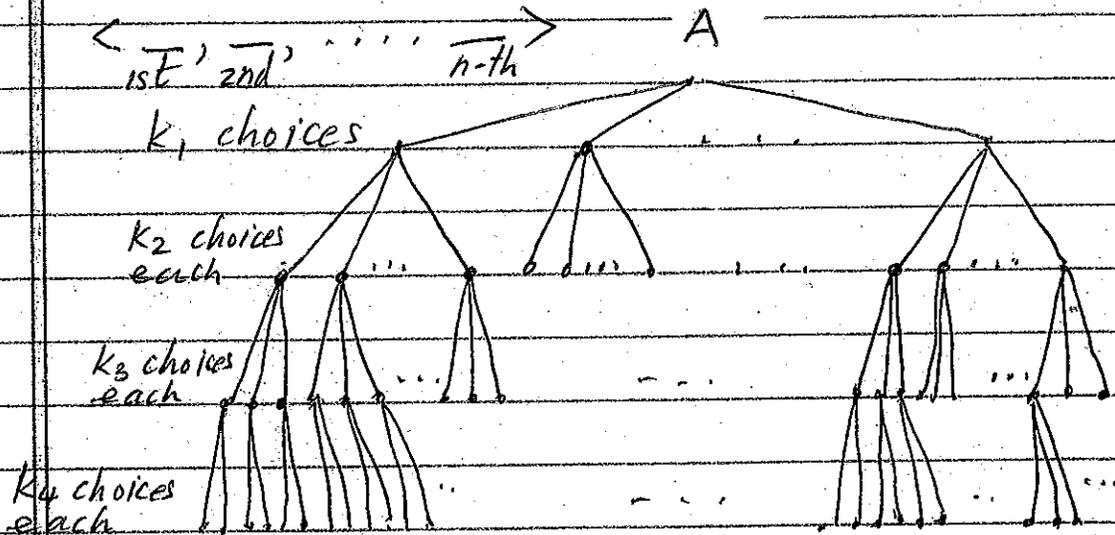
(c) Our answer will be  $|A \cup B|$ . And as in part (b)  
 $|A \cup B| = |A| + |B| - |A \cap B|$   
 $= \lfloor \frac{1000}{12} \rfloor + \lfloor \sqrt{1000} \rfloor - \lfloor \frac{\sqrt{1000}}{6} \rfloor = 83 + 31 - 5 = 109$ .

(d) Our answer will be  $|A^c \cap C^c| = |(A \cup C)^c| = |U| - |A \cup C| = 833$   
 or  $|U| - |A| - |C| + |A \cap C| = 1000 - 83 - 100 + 16 = 1000 - 167 = 833$ .

# The Multiplication Principle

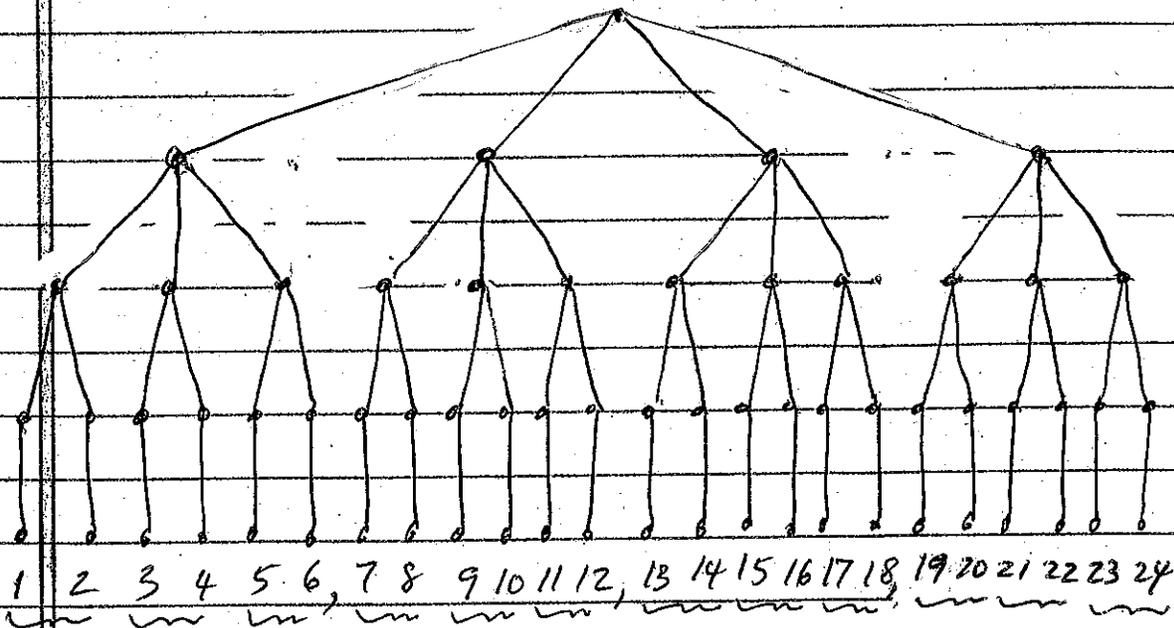
If  $A$  is a set of  $n$ -tuples and there are  $k_i$  ways of choosing the  $i$ -th component of the elements of  $A$  (for  $i=1, \dots, n$ ), then

$$|A| = k_1 \cdot k_2 \cdot k_3 \cdot \dots \cdot k_n$$



Ex.3 Let us illustrate how big the set  $A$  will be when  $n=4$  and  $k_1=4, k_2=3, k_3=2$ , and  $k_4=1$ .

As expected, we get  $4(3)(2)(1) = 24$  elements in  $A$



Ex 4 How many 3-digit (base 10) numerals

(9)

(a) have only even digits?

(b) begins with an even digit & ends with an odd digit?

(c) have all their digits distinct (different)?

(d) are even & have all their digits distinct?

Sol. (a)  $\langle \underbrace{4}_{\substack{\uparrow \\ \text{cannot be } 0}}, \underbrace{5}_{\substack{\uparrow \\ \text{can be } 0, 2, 4, 6, 8}}, \underbrace{5}_{\substack{\uparrow \\ \text{can be } 0, 2, 4, 6 \text{ or } 8}} \rangle$   
 So our answer is  $4(5)(5) = 100$ .

(b)  $\langle \underbrace{4}_{\substack{\uparrow \\ \text{can be } 2, 4, 6, \text{ or } 8}}, \underbrace{10}_{\substack{\uparrow \\ \text{can be any of the } 10}}, \underbrace{5}_{\substack{\uparrow \\ \text{can be } 1, 3, 5, 7, 9}} \rangle$   
 So our answer is  $4(10)(5) = 200$ .

(c)  $\langle \underbrace{9}_{\substack{\uparrow \\ \text{cannot be } 0}}, \underbrace{9}_{\substack{\uparrow \\ \text{cannot be first digit}}}, \underbrace{8}_{\substack{\uparrow \\ \text{cannot be 1st or 2nd digit}}} \rangle$   
 So our answer is  $9(9)(8) = 81(8) = 648$ .

(d) First observe that a numeral is even if and only if it ends in an even digit. Now split the problem into two cases.

Case (i): numeral ends in 0:  $\langle \underbrace{9}_{\substack{\uparrow \\ \text{cannot be } 0}}, \underbrace{8}_{\substack{\uparrow \\ \text{cannot be } 0 \text{ or the 1st digit}}}, \underbrace{1}_{\substack{\uparrow \\ \text{only } 0}} \rangle$

Case (ii) numeral ends in 2, 4, 6 or 8:  $\langle \underbrace{8}_{\substack{\uparrow \\ \text{choose this 2nd; not } 0 \text{ \& not } 3rd \text{ component}}}, \underbrace{8}_{\substack{\uparrow \\ \text{choose this first}}}, \underbrace{4}_{\substack{\uparrow \\ \text{choose this last - anything except 3rd \& 1st comp.}}} \rangle$   
 So our answer will be:

$$9(8)(1) + 8(8)(4) = 72 + 256 = 328.$$

## §2 Permutations & Combinations of sets.

Def. Let  $A = \{a_1, a_2, \dots, a_n\}$  be a set of  $n$  elements and  $r$  be an integer with  $0 \leq r \leq n$ . An  $r$ -permutation of  $A$  is an  $r$ -tuple of  $r$  distinct elements of  $A$ . When  $r=n$ , we usually call an  $n$ -permutation of  $A$  just a permutation of  $A$ .

Note: Since an  $r$ -tuple of elements of  $A$  is just a function from  $\{1, 2, 3, \dots, r\}$  to  $A$ , an  $r$ -tuple of distinct elements of  $A$  would be an injective function from  $\{1, 2, 3, \dots, r\}$  to  $A$ . So an  $r$ -permutation is just an injection from  $\{1, 2, \dots, r\}$  to  $A$  and we can write it as:  $(1 \ 2 \ 3 \ \dots \ r)$   
 $(a_{i_1} \ a_{i_2} \ a_{i_3} \ \dots \ a_{i_r})$   
 where the  $a_{i_k}$ 's are distinct.

Def. An  $r$ -combination of  $A$  is a subset of  $A$  containing  $r$  elements. We sometimes call an  $r$ -combination of  $A$  an  $r$ -subset of  $A$ .

Ex. 1(a) Let  $A = \{a, b, c\}$ . Then the set of all 2-permutations of  $A$  is

$$\{\langle a, b \rangle, \langle a, c \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, a \rangle, \langle c, b \rangle\}$$

We usually abbreviate  $\langle a, b \rangle$  by  $ab$  and so on.

So this set is  $\{ab, ac, ba, bc, ca, cb\}$ .

(b) The set of all permutations of  $A$  is

$$\{\langle a, b, c \rangle, \langle a, c, b \rangle, \langle b, a, c \rangle, \langle b, c, a \rangle, \langle c, a, b \rangle, \langle c, b, a \rangle\}$$

$$= \{abc, acb, bac, bca, cab, cba\}$$

Ex. 1 (c) The set of all 2-combinations of  $A$  is  $\{\{a,b\}, \{a,c\}, \{b,c\}\}$  and the set of all 3-combinations of  $A$  is  $\{\{a,b,c\}\}$ . (11)

Ex. 2 The set of all 0-permutations of  $A$  is  $\{\langle \rangle\}$  where  $\langle \rangle$  is the empty sequence. The set of all 0-combinations of  $A$  is  $\{\emptyset\}$ .

Prop. 1 Let  $P(n,r)$  be the number of  $r$ -permutations of  $\{1,2,3,\dots,n\}$ . Then  $P(n,r) = \frac{n!}{(n-r)!}$

Proof: An  $r$ -permutation of  $\{1,2,\dots,n\}$  is an  $r$ -tuple of  $r$  distinct elements of  $\{1,2,\dots,n\}$ . Now there are  $n$  ways of choosing the first component,  $(n-1)$  ways of choosing the second component,  $(n-2)$  ways of choosing the 3rd component

$\vdots$   
 $(n-(r-1))$  ways of choosing the  $r$ -th component.

So by the multiplication principle,

$\left\langle \begin{array}{cccc} \text{1st} & \text{2nd} & \text{3rd} & \text{r-th comp} \\ \hline n \text{ choices} & (n-1) \text{ choices} & (n-2) \text{ ch} & (n-(r-1)) \text{ ch.} \end{array} \right\rangle$

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\dots[n-(r-1)] \\ &= n(n-1)(n-2)\dots[n-(r-1)] \cdot (n-r)! / (n-r)! \\ &= n! / (n-r)! \end{aligned}$$

Prop. 2 Let  $C(n,r)$  be the number of  $r$ -combinations of  $\{1,2,3,\dots,n\}$ . Then  $C(n,r) = \frac{n!}{r!(n-r)!}$

Proof: Consider an  $r$ -combination of  $\{1,2,3,\dots,n\}$ .

This  $r$ -combination can be ordered in  $P(n, r) = \binom{n}{r} r!$  ways to produce  $r!$   $r$ -permutations of  $\{1, 2, 3, \dots, n\}$ . Since each  $r$ -permutation of  $\{1, 2, 3, \dots, n\}$  can be obtained by ordering a unique  $r$ -combination, it follows that

$$P(n, r) = (r!) \cdot C(n, r).$$

$$\text{Thus } C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Notation: We shall use the expression  $\binom{n}{r}$  to denote  $\frac{n!}{r!(n-r)!}$  for any  $n \in \mathbb{N}$  and  $r$  with  $0 \leq r \leq n$ .

When  $r > n$ , we will take  $\binom{n}{r}$  to be 0. The expression  $\binom{n}{r}$  is pronounced as "n choose r".

- Ex. 3 How many 3-subsets of  $\{1, 2, 3, \dots, 20\}$  have
- exactly one odd element?
  - at most one odd element?
  - at least one odd element?

Sol. (a) We need one odd element of  $\{1, 2, \dots, 20\}$  — and there are  $\binom{10}{1}$  ways of choosing this odd element. We also need 2 even elements of  $\{1, 2, \dots, 20\}$  — and there are  $\binom{10}{2}$  ways of choosing these 2 even elements. So our answer is  $\binom{10}{1} \cdot \binom{10}{2} = 10 \cdot \frac{10 \cdot 9}{2 \cdot 1} = 10(45) = 450$ .

(b) Answer = no. of 3-subsets with 0 odd elements + no. of 3-subsets with 1 odd element

$$= \binom{10}{0} \binom{10}{3} + \binom{10}{1} \binom{10}{2} = 1 \cdot \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} + 450 = 120 + 450 = 570$$

Ex. 3(b) Answer = No. of 3-subsets with 1 odd element  
 + No. of 3-subsets with 2 odd elements  
 + No. of 3-subsets with 3 odd elements  
 $= \binom{10}{1} \binom{10}{2} + \binom{10}{2} \binom{10}{1} + \binom{10}{3} \binom{10}{0} = 450 + 450 + 120 = 1020.$

Alt. answer = No. of 3-subsets of  $\{1, 2, 3, \dots, 20\}$   
 - No. of 3-subsets with 0 odd elements  
 $= \binom{20}{3} - \binom{10}{0} \binom{10}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} - 120 = 1140 - 120 = 1020.$

- Ex. 4. How many 3-permutations of  $\{1, 2, \dots, 20\}$  have
- (a) exactly one odd term?
  - (b) at most one odd term?
  - (c) at least one odd term?

Sol. Since a 3-subset of  $\{1, 2, \dots, 20\}$  will produce  $3! = 6$  3-permutations of  $\{1, 2, 3, \dots, 20\}$  our answers will just be 6 times the corresponding answers in Ex. 3. So our answers are:

(a)  $6(450) = 2700$ , (b)  $6(570) = 3420$ , (c)  $6(1020) = 6120.$

Ex 5 In how many ways can place 8 rooks on an  $8 \times 8$  chess-board so that no two rooks attack each other (i.e., so that no two rooks are in the same row or same column)

Sol. Let  $\langle i, c_i \rangle$  be the position of the single rook in row  $i$ . Here  $c_i$  denotes the column in which the rook is placed in row  $i$ . Now an arrangement of 8 rooks on the chessboard

with no two rooks attacking each other will be an 8-tuple: injective function  $f: \{1, 2, \dots, 8\} \rightarrow \{1, 2, \dots, 8\}$

$$f = \{(1, c_1), (2, c_2), (3, c_3), \dots, (8, c_8)\}$$

where  $c_1, c_2, c_3, \dots, c_8$  are all distinct.

So  $\langle c_1, c_2, \dots, c_8 \rangle$  will be a permutation of  $\{1, 2, 3, \dots, 8\}$ . Since there are  $8!$  permutations of  $\{1, 2, \dots, 8\}$ , there will be  $8!$  ways of arranging the 8 rooks in mutually non-attacking positions.

Def. A subsequence of the sequence  $\langle a_1, a_2, \dots, a_n \rangle$  is any sequence of the form  $\langle a_{i_1}, a_{i_2}, \dots, a_{i_k} \rangle$  where  $1 \leq i_1 < i_2 < i_3 < \dots < i_k \leq n$ .

Ex. 6 Find all the subsequences of  $\langle a_1, a_2, a_3, a_4 \rangle$  of lengths 0, 1, and 2 respectively

Sol. (a) There is only one subsequence of length 0, namely  $\langle \rangle$ .

(b) There are 4 subsequences of length 1, which corresponds to  $i_1=1, i_1=2, i_1=3$  &  $i_1=4$ . These are  $\langle a_1 \rangle, \langle a_2 \rangle, \langle a_3 \rangle$ , and  $\langle a_4 \rangle$

(c) There are 6 subsequences of length 2 and these correspond to:

$i_1=1 \ \& \ i_2=2$	$i_1=1 \ \& \ i_2=3$	$i_1=1 \ \& \ i_2=4$
$\langle a_1, a_2 \rangle$	$\langle a_1, a_3 \rangle$	$\langle a_1, a_4 \rangle$

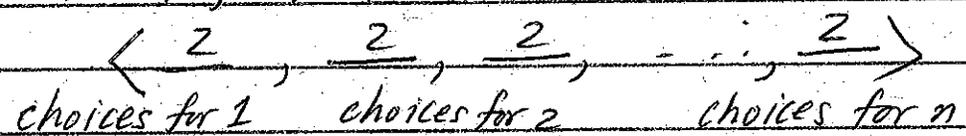
$i_1=2 \ \& \ i_2=3$	$i_1=2 \ \& \ i_2=4$	$i_1=3 \ \& \ i_2=4$
$\langle a_2, a_3 \rangle$	$\langle a_2, a_4 \rangle$	$\langle a_3, a_4 \rangle$

Prop 3  $\langle a_1, a_2, \dots, a_n \rangle$  has  $\binom{n}{k}$  subsequences of length  $k$ .

Proof: Each subsequence  $\langle a_{i_1}, a_{i_2}, \dots, a_{i_k} \rangle$  of length  $k$  of  $\langle a_1, a_2, \dots, a_n \rangle$  corresponds to a  $k$ -subset  $\{i_1, i_2, \dots, i_k\}$  of  $\{1, 2, \dots, n\}$  with the  $i_1, \dots, i_k$  listed in increasing order. Since there are  $\binom{n}{k}$   $k$ -subsets of  $\{1, 2, \dots, n\}$ , it follows that there are  $\binom{n}{k}$  subsequences of  $\langle a_1, \dots, a_n \rangle$  of length  $k$ .

Prop 4 The total number of subsets of  $\{1, 2, \dots, n\}$  is  $2^n$ . Consequently there are  $2^n$  possible subsequences of  $\langle a_1, a_2, \dots, a_n \rangle$ .

Proof (a) A subset of  $\{1, 2, 3, \dots, n\}$  is determined by  $n$  choices we make: Is  $i \in$  the subset for  $i = 1, 2, \dots, n$ .



Now there are 2 choices for 1 (either it is in the subset or it is out), 2 choices for 2, ..., and 2 choices for  $n$ . So by the multiplication principle, it follows that there are  $2^n$  possible subsets.

(b) We know that a subsequence of  $\langle a_1, \dots, a_n \rangle$  corresponds to a subset of  $\{1, 2, \dots, n\}$ . Since there are  $2^n$  possible subsets of  $\{1, 2, \dots, n\}$  there will be  $2^n$  possible subsequences of  $\langle a_1, \dots, a_n \rangle$ .

### §3. Inversion sequence of a permutation:

Ex.1 Consider the permutation  $\sigma = \langle 3, 2, 5, 1, 4 \rangle$  of  $\{1, 2, 3, 4, 5\}$ . The ordered pair  $(3, 1)$  is called an inversion of  $\sigma$  because in their natural order 1 precedes 3 — but in  $\sigma$ , 3 precedes 1. Similarly  $(2, 1)$ ,  $(5, 1)$ ,  $(3, 2)$ , &  $(5, 4)$  are inversions of  $\sigma$ .  $(3, 4)$  is not an inversion in  $\sigma$ .

Def. Let  $\sigma = \langle a_1, a_2, \dots, a_n \rangle$  be a permutation of  $\{1, 2, \dots, n\}$ . An inversion of  $\sigma$  is any ordered pair  $(a_i, a_j)$  with  $i < j$  &  $a_i > a_j$ .

The number of inversions of  $\sigma$  with respect to the integer  $k$  ( $1 \leq k \leq n$ ) is defined by  $i_k(\sigma) =$  number of elements which precede  $k$  in  $\sigma$  and are bigger than  $k$ .

The inversion sequence of the permutation  $\sigma$  is the sequence  $\langle i_1(\sigma), i_2(\sigma), \dots, i_n(\sigma) \rangle$ .

Ex.2 Let  $\sigma = \langle 3, 2, 5, 1, 4 \rangle$ . Then  $i_1(\sigma) = 3$ ,  $i_2(\sigma) = 1$ ,  $i_3(\sigma) = 0$ ,  $i_4(\sigma) = 1$  and  $i_5(\sigma) = 0$ . So the inversion sequence of  $\sigma$  is  $\langle 3, 1, 0, 1, 0 \rangle$ .

Note: Since there are only  $n-k$  elements of  $\{1, 2, \dots, n\}$  that are greater than  $k$ , it follows that  $0 \leq i_k(\sigma) \leq n-k$  for any permutation  $\sigma$  of  $\{1, \dots, n\}$ .

Qn: Suppose  $(a_1, a_2, \dots, a_n)$  is a sequence of integers with  $0 \leq a_k \leq n-k$  for  $k=1, \dots, n$  is it always true that  $(a_1, \dots, a_n)$  is the inversion sequence of some permutation  $\sigma$  of  $\{1, 2, 3, \dots, n\}$ ?

Ans: Yes.

Theorem 4: Let  $(a_1, \dots, a_n)$  be a sequence of integers with  $0 \leq a_k \leq n-k$  for  $k=1, 2, \dots, n$ . Then there is a unique permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  such that  $(i_1(\sigma), \dots, i_n(\sigma)) = (a_1, \dots, a_n)$ .

Proof: We shall describe an algorithm for finding the unique permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  such that  $(i_1(\sigma), \dots, i_n(\sigma)) = (a_1, \dots, a_n)$ .

Step 1: First write down  $n$  to get a sequence  $(n)$

Step 2: Then insert  $n-1$  in this sequence so that there are  $a_{n-1}$  bigger terms in front of  $n-1$

Step 3: Then insert  $n-2$  in the sequence obtained from step 2, so that there are  $a_{n-2}$  bigger terms in front of  $n-2$ .

Step  $n-(k-1)$ : In general insert  $k$  in the sequence so that there are  $a_k$  bigger terms in front of  $k$

Step  $n$ : In the last step we will insert 1 in the sequence so that there are  $a_1$  bigger terms in front of 1.

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If we proceed as in this algorithm we will get a permutation  $\sigma$  of  $\{1, 2, 3, \dots, n\}$  with  $\langle i_1(\sigma), i_2(\sigma), \dots, i_n(\sigma) \rangle = \langle a_1, a_2, \dots, a_n \rangle$ .

Ex. 3 Let  $\langle a_1, a_2, a_3, a_4, a_5 \rangle = \langle 3, 1, 0, 1, 0 \rangle$ . Then for each  $k$ ,  $0 \leq a_k \leq n-k$ . Find the permutation  $\sigma$  of  $\{1, 2, \dots, 5\}$  which has inversion sequence  $\langle 3, 1, 0, 1, 0 \rangle$ .

Sol. 1. Write down  $\langle 5 \rangle$

2. Insert 4, so that there are  $a_4 = 1$  bigger terms in front of 4 to get  $\langle 5, 4 \rangle$

3. Insert 3, so that there are  $a_3 = 0$  bigger terms in front of 3 to get  $\langle 3, 5, 4 \rangle$

4. Insert 2, so that there are  $a_2 = 1$  bigger terms in front of 2 to get  $\langle 3, 2, 5, 4 \rangle$

5. Insert 1, so that there are  $a_1 = 3$  bigger terms in front of 1 to get  $\langle 3, 2, 5, 1, 4 \rangle$ .

So  $\sigma = \langle 3, 2, 5, 1, 4 \rangle$

Def. Let  $\sigma$  be a permutation of  $\{1, 2, 3, \dots, n\}$ . The total no. of inversions in  $\sigma$  is defined by

$$I_T(\sigma) = i_1(\sigma) + i_2(\sigma) + \dots + i_{n-1}(\sigma) + i_n(\sigma).$$

Ex. 4 Find the number of permutations of  $\{1, 2, 3, \dots, 7\}$  in which the total no. of inversions is:

(a) 21, (b) 20, (c) 19, (d) 18.

Ex.4 (a) The maximum no. of inversions will come <sup>(19)</sup> from the permutation of  $\{1, 2, \dots, n\}$  with inversion sequence  $\langle 6, 5, 4, 3, 2, 1, 0 \rangle$ . The permutation which corresponds to this inversion sequence is  $\langle 7, 6, 5, 4, 3, 2, 1 \rangle$ . So the no. of permutations with 21 inversions is 1.

(b) To get a total no. of inversions of 20, we need to reduce exactly one of the first 6 terms of  $\langle 6, 5, 4, 3, 2, 1, 0 \rangle$  by 1. Since there are  $\binom{6}{1}$  ways of doing this there will be  $\binom{6}{1} = 6$  permutations of  $\{1, 2, \dots, 7\}$  with 20 inversions.

(c) To get a total no. of inversions of 19, we need to reduce exactly 2 of the first 6 terms of  $\langle 6, 5, 4, 3, 2, 1, 0 \rangle$  by 1 or to reduce exactly 1 of the first 5 terms of  $\langle 6, 5, 4, 3, 2, 1, 0 \rangle$  by 2. Since there are  $\binom{6}{2}$  ways of choosing two of the first 6 terms and  $\binom{5}{1}$  ways of choosing one of the first 5 terms, there will be  $\binom{6}{2} + \binom{5}{1} = 15 + 5 = 20$  permutations of  $\{1, 2, \dots, 7\}$  with 19 inversions.

(d) Reduce 3 of the first 6 terms by 1; reduce 1 of the first 4 terms by 3; or reduce one of the first 5 terms by 2 & one of the other 5 (of 6 terms) by 1. So answer =  $\binom{6}{3} + \binom{4}{1} + \binom{5}{1} \cdot \binom{5}{1} = 20 + 4 + 25 = 49$ .