Solving linear ODEs with discontinuous non-homog. terms

Ex. 2 Find the solution of the ODE below by using Laplace transforms. $y'(t) - 3\,y(t) = 6\,u_2(t)$ with $y(0) = 4$.

Sol. $\mathcal{L}\{y'\} - 3\,\mathcal{L}\{y\} = 6\,\mathcal{L}\{u_2(t)\}$

$\therefore\; s\,\mathcal{L}\{y\} - y(0) - 3\,\mathcal{L}\{y\} = 6 . e^{-2s}/s$

$\therefore\; (s-3)\,\mathcal{L}\{y\} = y(0) + 6 . e^{-2s}/s = 4 + 6e^{-2s}/s$

$$\therefore\; \mathcal{L}\{y\} = \frac{4}{s-3} + \frac{6\,e^{-2s}}{(s-3)\,s}$$

$$= \frac{4}{s-3} + \frac{2e^{-2s}}{s-3} - \frac{2e^{-2s}}{s}$$

$$\therefore\; y(t) = 4 . \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - 2 . \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} + 2 . \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-3}\right\}$$

$$= 4 . e^{3t} - 2 . u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t-2) + 2 . u_2(t) . \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t-2)$$

$$= 4 . e^{3t} - 2 . u_2(t) . 1 + 2 . u_2(t) . e^{3(t-2)}$$

$$= \begin{cases} 4e^{3t} - 2(0) + 2(0) & 0 \le t \le 2 \\ 4e^{3t} - 2 + 2e^{3(t-2)} & t > 2 \end{cases}$$

$$= \begin{cases} 4e^{3t} & 0 \le t \le 2 \\ 4e^{3t} - 2 + 2 . e^{-6} . e^{3t} & t > 2 \end{cases}$$

END.

Extras: **Check**: $y(2-) = 4e^{3(2)} = 4e^{6}$

$y(2+) = 4e^{3(2)} - 2 + 2e^{-6} . e^{3(2)} = 4e^{6}$

Note: $\mathcal{L}\{u_a(t) . f(t-a)\}(s) = e^{-as} . \overbrace{\mathcal{L}\{f\}(s)}^{F(s)}$

$\therefore\; \mathcal{L}^{-1}\{e^{-as} . F(s)\}(t) = u_a(t) . \mathcal{L}^{-1}\{F\}(t-a)$

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 = f(t)$, $f(t-2) = 1$; $\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t} = f(t)$, $f(t-2) = e^{3(t-2)}$