MAP 2302 - Differential Equations   Florida Int'l Univ.

Test #1 - Fall 2006   TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. No CALCULATORS or formula sheets are allowed.

(15) 1. Find the solution of the initial value problem

$$\frac{dy}{dx} - 3y = 6e^{x} \quad \text{with} \quad y(0) = 2.$$

(15) 2. Find the general solution of the ODE

$$\frac{dy}{dx} - \frac{4}{x}y = 8x \cdot y^{3/2}.$$

(15) 3. (a) Define what it means for $dy/dx = f(x,y)$ to be homogeneous.

(b) Solve the ODE $\quad 3(x^3 - y^2x)\,dx + 2x^2y\,dy = 0$

(15) 4. The population of a colony satisfies the logistic law $dP/dt = P/2 - P^2/8000$, where t is measured in years. If $P(0) = 1000$, find the population after t years.

(20) 5. A box of mass 2kg is moving horizontally with velocity $5\,ms^{-1}$ when it lands on a horizontal surface. If the air resistance is λv where $\lambda = 1\,kgs^{-1}$ and the coefficient of friction μ is $1/4$,

(a) how long will the box slide on the surface, and

(b) how far will it slide?   [use $g = 10\,ms^{-2}$]

(20) 6. (a) Define what is the total differential of $F(x,y)$ and what is an integrating factor of the non-exact ODE

$M(x,y)\,dx + N(x,y)\,dy = 0$. (*)

(b) Find the solution of $(y^3x + y^2)\,dx + (y^4 - xy)\,dy = 0$.

MAP 2302 - Differential Eq. Florida Int'l Univ.

Solutions to Test #1 Fall 2006

1. $dy/dx - 3y = 6e^x$. This is a linear ODE, so the integrating factor $= e^{\int p(x)dx} = e^{\int -3dx} = e^{-3x}$.

$\therefore \quad e^{-3x}(dy/dx) - 3.e^{-3x}.y = 6e^x.e^{-3x}$

$\therefore \quad \frac{d}{dx}(y.e^{-3x}) = 6e^{-2x}$

$\therefore \quad y.e^{-3x} = -3e^{-2x} + C$

But $y(0) = 2$, so $2.e^{-0} = -3.e^{-0} + C$

$\therefore \quad C = 5$. Hence $y.e^{-3x} = -3e^{-2x} + 5$

$\therefore \quad y = 5e^{3x} - 3e^{-2x}.e^{3x} = 5e^{3x} - 3e^x$.

2. (a) The ODE $dy/dx = f(x,y)$ is homogeneous if we can find a function g such that $f(x,y) = g\left(\frac{y}{x}\right)$.

(b) Here $2x^2y\,dy = 3(y^2x - x^3)dx$. So

$dy/dx = \frac{3(y^2x - x^3)}{2x^2y} = \frac{3}{2}\frac{y}{x} - \frac{3}{2}\frac{x}{y}$

So this ODE is homog. Put $y = xv$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = y/x$. Hence

$v + x\frac{dv}{dx} = \frac{3}{2}\frac{y}{x} - \frac{3}{2}\frac{x}{y} = \frac{3}{2}v - \frac{3}{2}\frac{1}{v}$

$\therefore \quad x\frac{dv}{dx} = \frac{1}{2}v - \frac{3}{2}.\frac{1}{v} = \frac{1}{2}.\frac{v^2 - 3}{v}$

$\therefore \quad \frac{2v\,dv}{v^2 - 3} = \frac{dx}{x}$

So $\ln(v^2 - 3) = \ln x + C$

$\therefore \quad v^2 - 3 = e^C.x = Ax$

$\therefore \quad v^2 = Ax + 3$. So $\frac{y^2}{x^2} = Ax + 3$

Hence $y^2 = x^2(Ax + 3)$.

2. This is a Bernoulli ODE. So put $v = y^{1-n} = y^{1-3/2}$

Then $v = y^{-1/2}$ and $dv/dx = (-1/2)\, y^{-3/2}$. Now multiply the original ODE by $(1-3/2) \cdot y^{-3/2} = (-\frac{1}{2}) y^{-3/2}$

Then $-\frac{1}{2} \cdot y^{-3/2} \cdot \frac{dy}{dx} - \frac{4}{x} \cdot (-\frac{1}{2}) y^{-3/2} \cdot y = -\frac{1}{2} \cdot 8x \cdot y^{-3/2} \cdot y^{3/2}$

$\therefore \quad -\frac{1}{2} \cdot y^{-3/2} \frac{dy}{dx} + \frac{2}{x} y^{-1/2} = -4x$

$\therefore \quad \frac{dv}{dx} + \frac{2}{x} v = -4x$

This is a linear ODE with integr. factor $= e^{\int p(x)dx}$

$= e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln(x^2)} = x^2$.

$\therefore \quad x^2 \frac{dv}{dx} + x^2 \cdot \frac{2}{x} v = -4 \cdot x^2 \cdot x$

$\therefore \quad \frac{d}{dx}(x^2 v) = -4x^3 \Rightarrow x^2 v = -x^4 + C$

$\therefore \quad v = \frac{C - x^4}{x^2}$. $\therefore \quad y = \frac{1}{v^2} = \left(\frac{x^2}{C - x^4}\right)^2 = \frac{x^4}{(C - x^4)^2}$

4. We have $\frac{dP}{dt} = \frac{P}{2} - \frac{P^2}{8000} = \frac{P(4000-P)}{2(4000)}$

$\therefore \quad \frac{4000\, dP}{P(4000-P)} = \frac{dt}{2}$

$\therefore \quad \left(\frac{1}{P} + \frac{1}{4000-P}\right) dP = \frac{dt}{2}$

$\therefore \quad \ln P - \ln(4000-P) = \frac{t}{2} + C$

$\therefore \quad \ln\left(\frac{P}{4000-P}\right) = \frac{t}{2} + C$

$\therefore \quad \frac{P}{4000-P} = e^C \cdot e^{t/2} = A \cdot e^{t/2}$

But $P(0) = 1000$. So $\frac{1000}{4000-1000} = A \cdot e^0$

$\therefore \quad A = \frac{1}{3}$.

$\therefore \quad \frac{P}{4000-P} = \frac{1}{3} e^{t/2}$. $\therefore \quad 3P = (4000-P) e^{t/2}$

$\therefore \quad P(3 + e^{t/2}) = 4000 e^{t/2}$. $\therefore \quad P = \frac{4000 e^{t/2}}{3 + e^{t/2}}$

$\therefore \quad P(t) = \frac{4000}{1 + 3e^{-t/2}}$.

5. We have $m\frac{dv}{dt} = -\text{resistance} - \text{friction}$, because friction & resistance work against the motion. So

$$2\frac{dv}{dt} = -\lambda . v - \mu m g = -1.v - \frac{1}{4}.2.10$$

$$\therefore\ 2\frac{dv}{dt} = -(v+5). \qquad \therefore\ \frac{dv}{v+5} = -\frac{dt}{2}$$

$$\therefore\ \ln(v+5) = -\frac{t}{2} + C_1$$

$$\therefore\ v+5 = e^{C_1}.e^{-t/2} = A.e^{-t/2}$$

$$\therefore\ v = -5 + Ae^{-t/2}$$

But $v(0) = 5$. So $5 = -5 + A.e^0$

$$\therefore\ A = 10. \qquad \therefore\ v(t) = -5 + 10e^{-t/2}$$

(a) The box will slide until $v(t) = 0$. So

$$0 = -5 + 10e^{-t/2}$$

$$\therefore\ 5 = 10e^{-t/2} \Rightarrow 5e^{t/2} = 10$$

$$\Rightarrow e^{t/2} = 2$$

$\therefore\ t/2 = \ln 2$. So $t = 2\ln 2$ seconds.

(b) We know that $v(t) = -5 + 10e^{-t/2}$

So $\frac{dx}{dt} = -5 + 10e^{-t/2}$

$$\therefore\ x(t) = -5t - 20e^{-t/2} + C_2$$

But $x(0) = 0$. So $0 = 0 - 20 + C_2$

$$\therefore\ C_2 = 20.$$

$$\therefore\ x(t) = 20 - 20e^{-t/2} - 5t$$

So the box will slide a total distance of

$$x(2\ln 2) = 20 - 20.e^{-2\ln 2/2} - 5.(2\ln 2)$$

$$= 20 - 20.\frac{1}{2} - 10\ln 2 \quad \text{bec. } e^{-\ln 2} = \frac{1}{2}.$$

$$= 10(1 - \ln 2) \text{ metres.}$$

6 (a) The total differential of $F(x,y)$ is defined by
$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$.
An integrating factor of the ODE (*) is any function $\mu(x,y)$ such that $\mu M dx + \mu N dy = 0$ is exact. (For this to happen we need $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$).

(b) Let $M = y^3x + y^2$ and $N = y^4 - xy$.
Then $\frac{\partial M}{\partial} - \frac{\partial N}{\partial x} = 3y^2x + 2y - (-y) = 3y^2x + 3y$
$= 3y(xy+1)$
So the ODE is not exact. But
$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3y(xy+1)}{y^2(xy+1)} = \frac{3}{y} = g(y)$
So $e^{-\int g(y)dy}$ will be an integrating factor of the ODE.
$e^{-\int g(y)dy} = e^{-\int \frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3} = \frac{1}{y^3}$
$\therefore \frac{1}{y^3}\cdot(y^3x + y^2)dx + \frac{1}{y^3}(y^4 - xy)dy$
$= \underbrace{\left(x + \frac{1}{y}\right)}_{M_2}dx + \underbrace{\left(y - \frac{x}{y^2}\right)}_{N_2}dy = 0$ is exact.

$\frac{\partial F}{\partial x} = M_2 = x + \frac{1}{y} \Rightarrow F = \frac{x^2}{2} + \frac{x}{y} + \varphi(y)$
$\therefore \frac{\partial F}{\partial y} = 0 + x\cdot\left(-\frac{1}{y^2}\right) + \varphi'(y) = -\frac{x}{y^2} + \varphi'(y)$
But $\frac{\partial F}{\partial y} = N_2 = y - \frac{x}{y^2}$. $\therefore \varphi'(y) = y$
$\therefore \varphi(y) = \frac{y^2}{2} + C_1$. Thus $F = \frac{x^2}{2} + \frac{x}{y} + \frac{y^2}{2} + C_1$
Since $dF = M_2dx + N_2dy = 0$, $F = C_2$
$\therefore C_2 = \frac{x^2}{2} + \frac{x}{y} + \frac{y^2}{2} + C_1$
$\therefore \frac{x^2}{2} + \frac{x}{y} + \frac{y^2}{2} = C_2 - C_1 = C$
Solution is $\frac{x^2}{2} + \frac{y^2}{2} + \frac{x}{y} = C$.