MAP 2302 - Differential Equations — Florida Int'l Univ.

TEST #2 — Fall 2006 — TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. NO CALCULATORS OR FORMULA SHEETS ARE ALLOWED.

(15) 1. Find the solution of each of the following ODEs

(a) $y'' + 9y = 0$ with $y(0) = 5$ and $y'(0) = -6$.

(b) $y'' - 4y' + 4y = 0$ with $y(0) = 4$ and $y'(0) = 5$

(20) 2. Find the general solution of each of the following ODEs

(a) $y'' - 4y = 12e^{2x}$ (b) $y'' + y' - 2y = -4x^2$.

(20) 3. Find y_c and give the minimal form of the y_p that one should try for each of the following ODEs

(a) $(D^2 - 4D + 5)^3 y = 5x \cdot e^{2x} \cdot \cos x$.

(b) $(D^2+1)^2 (D-3)^2 y = 3\sin x + 2\sin^2 x$.

(15) 4(a) Define what it means for $\{f_1, f_2, f_3\}$ to be linearly independent.

(b) Given that $f(x) = x$ is a solution of $(x^3+1)y'' - 6x^2 y' + 6xy = 0$ find a second linearly independent solution.

(15) 5.(a) Define what is the Wronskian of $\{f_1, f_2, f_3\}$.

(b) Find a particular solution of the ODE $y'' + y = \csc(x)$.

(15) 6. A body of mass 5 kg is attached to a Hooke-type spring and suspended from the ceiling. The natural length L is 6 m, the spring constant k is $25\ Nm^{-1}$, and the air resistance is λv where $\lambda = 30\ Nsm^{-1}$. If the spring is stretched by an amount of 3 m at time $t=0$, find the position of the body at all subsequent times. [Use $g = 10\ ms^{-2}$] At time $t=0$, the body is let loose from rest.

MAP 2302 - Differential Equations Florida Int'l Univ.

Solutions to Test #2 Fall 2006

1(a) $y'' + 9y = 0 \quad \therefore (D^2+9)y = 0$

$\therefore D^2+9 = 0$ (Aux. Eq), so $D = \pm 3i$

$\therefore y = C_1 \cos(3x) + C_2 \sin(3x)$

$y' = -3C_1 \sin(3x) + 3C_2 \cos(3x)$

$y(0) = 5 \Rightarrow 5 = C_1 . 1 + C_2 . 0 \Rightarrow C_1 = 5$

$y'(0) = -6 \Rightarrow -6 = -3C_1 . 0 + 3C_2 \Rightarrow C_2 = -2$

Hence $y = 5\cos(3x) - 2\sin(3x)$.

(b) $y'' - 4y' + 4y = 0 \quad \therefore (D^2 - 4D + 4)y = 0$

Aux. Eq. is $D^2 - 4D + 4 = 0 \Rightarrow (D-2)^2 = 0 \Rightarrow D = 2$ (twice)

$\therefore y = (C_1 + C_2 x) . e^{2x}. \quad \therefore y' = [2(C_1 + C_2 x) + C_2] e^{2x}$

$y(0) = 4 \Rightarrow 4 = (C_1 + C_2 . 0) . e^0 \Rightarrow C_1 = 4$

$y'(0) = 5 \Rightarrow 5 = 2(C_1 + C_2 . 0) + C_2 \Rightarrow C_2 = -3$

$\therefore y = (4 - 3x) . e^{2x}$.

2(a) Homog. Eq. is $y'' - 4y = 0. \quad \therefore (D^2 - 4)y = 0$

$\therefore (D-2)(D+2)y = 0 \Rightarrow y_c = C_1 e^{2x} + C_2 e^{-2x}$

Try $y_p = Ax . e^{2x}$. (because e^{2x} is part of y_c)

Then $y_p' = (A + 2Ax) e^{2x}$ and

$y_p'' = [2A + 2(A+2Ax)] e^{2x} = (4A + 4Ax) e^{2x}$

$\therefore y'' - 4y = 12e^{2x}$ becomes

$(4A + 4Ax) e^{2x} - 4 . Axe^{2x} = 12e^{2x}$

$\therefore 4Ae^{2x} = 12e^{2x} \Rightarrow A = 3$ because $e^{2x} \neq 0$

$\therefore y_p = 3 . xe^{2x}$. So

$y = y_c + y_p = C_1 e^{2x} + C_2 e^{-2x} + 3xe^{2x}$.

(b) Homog. Eq. is $y'' + y' - 2y = 0. \quad \therefore (D^2 + D - 2)y = 0$

$\therefore (D-1)(D+2)y = 0 \Rightarrow y_c = C_1 e^{x} + C_2 e^{-2x}$.

2 (b) Try $y_p = Ax^2 + Bx + C$ (bec. $-4x^2$ is a polynomial of degree 2). Then $y_p' = 2Ax + B$ and $y_p'' = 2A$.

So $y'' + y' - 2y = -4x^2$ becomes

$$2A + (2Ax+B) - 2(Ax^2+Bx+C) = -4x^2$$

$\therefore$ $-2A \cdot x^2 + (2A-2B) \cdot x + (2A+B-2C) = -4x^2$

$\therefore$ $-2A = -4 \Rightarrow A = 2$

$2A - 2B = 0 \Rightarrow B = A = 2$

$2A + B - 2C = 0 \Rightarrow C = \frac{1}{2}(2A+B) = 3$

$\therefore$ $y_p = 2x^2 + 2x + 3$. Hence

$$y = y_c + y_p = C_1 e^x + C_2 e^{-2x} + 2x^2 + 2x + 3.$$

3(a) $(D^2 - 4D + 5)y = 0 \Rightarrow D = [-(-4) \pm \sqrt{16-20}]/2 = 2 \pm i$ (3 times)

$\therefore$ $y_c = e^{2x}[(C_1 + C_2 x + C_3 x^2)\cos x + (C_4 + C_5 x + C_6 x^2)\sin x]$

The minimal form of the y_p that one should try is

$$y_p = e^{2x} \cdot x^3 \cdot [(A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x]$$

because $2+i$ is a root of the aux. eq. of multiplicity 3 and $5x$ is a polynomial of degree 1.

(b) $(D^2+1)^2(D-3)^2 y = 0 \Rightarrow D = 3$ (twice) or $D = \pm i$ (twice)

$\therefore$ $y_c = (C_1 + C_2 x)e^{3x} + (C_3 + C_4 x)\cos x + (C_5 + C_6 x)\sin x$.

Now $3\sin x + 2\sin^2 x = 3\sin x + 2 \cdot \frac{1}{2}(1 - \cos 2x)$. So

$(D^2+1)^2(D-3)^2 y = 3\sin x + 2\sin^2 x = 3\sin x + 1 - \cos(2x)$.

Since i is a root of the aux. eq. of multiplicity 2, we should try $(A_0 \cos x + B_0 \sin x) \cdot x^2$ as a part of the minimal y_p to cater for the "$3\sin x$". Also since "1" and "$-\cos(2x)$" are not part of the y_c, we also need to add $A_1 \cos(2x) + B_1 \sin(2x) + A_2$ to our y_p. Hence the minimal form of the y_p that one should try is

$$y_p = (A_0 \cos x + B_0 \sin x) \cdot x^2 + [A_1 \cos(2x) + B_1 \sin(2x)] + A_2.$$

4(a) The functions f_1, f_2, f_3 are linearly independent if $c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0 \Rightarrow c_1 = c_2 = c_3 = 0$.

(b) A second linearly indep. solution is given by $y = v.f(x)$ where

$$v = \int \frac{e^{\int -[a_1(x)/a_0(x)]dx}}{[f(x)]^2} dx = \int \frac{e^{-\int \frac{-6x^2}{x^3+1}dx}}{x^2} dx$$

$$= \int \frac{e^{2\ln(x^3+1)}}{x^2} dx = \int \frac{e^{\ln(x^3+1)^2}}{x^2} dx = \int \frac{(x^3+1)^2}{x^2} dx$$

$$= \int (x^4 + 2x + x^{-2}) dx = \frac{x^5}{5} + x^2 - x^{-1}$$

$$\therefore y_2(x) = v.f(x) = \left[\frac{x^5}{5} + x^2 - x^{-1}\right].x = \frac{x^6}{5} + x^3 - 1.$$

5(a) Wronskian of $f_1, f_2, f_3 = W[f_1, f_2, f_3] = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$

(b) We know one $y_p = v_1 y_1 + v_2 y_2$ where y_1 & y_2 are two linearly independent solutions of the homogeneous equation and

$$v_1' = \begin{vmatrix} 0 & y_2 \\ F(x) & y_2' \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \& \quad v_2' = \begin{vmatrix} y_1 & 0 \\ y_1' & F(x) \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Homog. Eq. is $y'' + y = 0$. So take $y_1 = \cos x$ & $y_2 = \sin x$.

Then $v_1' = \begin{vmatrix} 0 & \sin x \\ \csc x & \cos x \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \dfrac{-\csc x \sin x}{\sin^2 x + \cos^2 x} = \dfrac{-1}{1}$

$\therefore v_1 = \int -1 \, dx = -x.$

Also $v_2' = \begin{vmatrix} \cos x & 0 \\ -\sin x & \csc x \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \dfrac{\csc x . \cos x}{\sin^2 x + \cos^2 x}$

$$= \frac{1}{\sin x} . \frac{\cos x}{1} = \cot x$$

$\therefore v_2 = \int \cot x \, dx = -\ln(\csc x) = -\ln\left(\frac{1}{\sin x}\right) = \ln(\sin x)$

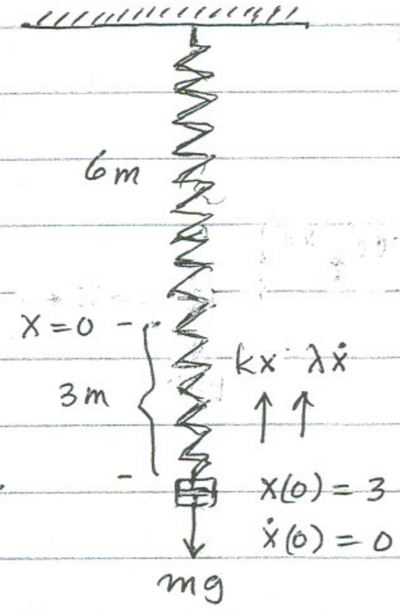
$\therefore y_p = v_1 y_1 + v_2 y_2$

$= -x . \cos x + \ln(\sin x) . \sin x$

$= -x . \cos x + \sin x . \ln(\sin x).$

6. Let $x(t)$ = amount the spring is extended at time t.

Then $\quad m\ddot{x} = F = mg - \lambda v - kx$



$\therefore \quad 5.\ddot{x} = 5.10 - 20.\dot{x} - 25.x$

$\therefore \quad \ddot{x} + 4\dot{x} + 5x = 10$

Homog. Eq. $(D^2+4D+5)x = 0$

$\therefore \quad D = (-4 \pm \sqrt{16-20})/2 = -2 \pm i$

$\therefore \quad x_c = e^{-2t}(A\cos t + B\sin t)$

Try $x_p = C$. Then $\dot{x}_p = 0$ & $\ddot{x}_p = 0$.

So $\quad \ddot{x} + 4\dot{x} + 5x = 10$ becomes

$\quad 0 + 0 + 5C = 10$

$\therefore \quad C = 2 \Rightarrow x_p(t) = 2$.

Hence $\quad x(t) = x_c(t) + x_p(t) = e^{-2t}(A\cos t + B\sin t) + 2$.

So $\quad \dot{x}(t) = -2e^{-2t}.(A\cos t + B\sin t)$

$\qquad + e^{-t}(-A\sin t + B\cos t) + 0$.

$x(0) = 3 \Rightarrow 3 = e^{-0}(A.1 + B.0) + 2 \Rightarrow A = 1$

$\ddot{x}(0) = 0 \Rightarrow 0 = -2.e^{-0}(A.1 + B.0) + e^{-0}(-A.0 + B)$

$\Rightarrow 0 = -2(1) + B \Rightarrow B = 2$

$\therefore \quad x(t) = e^{-2t}(\cos t + 2\sin t) + 2$ END

Note: When t is small, kx is positive and $\lambda\dot{x}$ is negative, so near the beginning the spring will be trying to pull up the mass while gravity & air resistance will be pulling downwards. $\dot{x}(t)$ will be positive when the mass is moving downwards.