MAP 2302 – Differential Equations Florida Int'l Univ.

TEST # 3 – Fall 2006 Time: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE. No calculators

(15) 1. Find the general solution of the ODE $4x^2y'' + y = 3\ln x$ by transforming it into a non-homogeneous constant coeff. linear equation in y & t.

(15) 2. Starting with $\mathcal{L}\{e^{at}\}(s) = 1/(s-a)$, use the theorems proved in class to find (a) $\mathcal{L}\{\sin t\}(s)$ and (b) $\mathcal{L}\{t^2 \cdot \sin t\}(s)$

(20) 3. Solve the following ODEs by using the Laplace Transform.

(a) $y'(t) + 2 \cdot y(t) = 6 \cdot e^t$ with $y(0) = 5$

(b) $y''(t) + 4 \cdot y'(t) + 5 \cdot y(t) = 0$ with $y(0) = 3$ & $y'(0) = -8$

(20) 4(a) Define what it means for x_0 to be a singular point and for x_0 to be a regular singular point of the ODE $y'' + P_1(x)y' + P_2(x)y = 0$.

(b) For each of the following ODEs, find the indicial equation and the form of two linearly indep. series solution about $x_0 = 0$.

(i) $2x^2y'' - xy' + xy = 0$ (ii) $4x^2y'' + (x+1)y = 0$

(15) 5. Find the first 5 non-zero terms of the power series solution of the ODE $y'' - xy' + 2y = 0$ with $y(0) = 1$ & $y'(0) = 3$.

(15) 6.(a) Find the indicial equation and the recurrence relation of the ODE $x^2y'' + (1/4 - x)y = 0$.

(b) Then find the first 4 non-zero terms of one non-trivial series solution about $x_0 = 0$.

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Solutions to Test #3 Fall 2006.

1. Put $x = e^t$. Then $t = \ln x$ and $xy' = \dot{y}$ & $x^2y'' = \ddot{y} - \dot{y}$.

So (*) $4x^2y'' + y = 3\ln x$ becomes

$$4(\ddot{y} - \dot{y}) + y = 3t. \quad \text{So } 4\ddot{y} - 4\dot{y} + y = 3t$$

$$\therefore \quad (4D^2 - 4D + 1)y = 3t \qquad (**)$$

Homog Eq. is $(4D^2 - 4D + 1)y = 0$. $\therefore (2D-1)^2 y = 0$

So $D = 1/2$ (twice). Hence $y_c = (C_1 + C_2 t)e^{t/2}$.

Try $y_p = At + B$. Then $\dot{y} = A$ & $\ddot{y} = 0$

So (**) becomes $4.(0) - 4.A + (At + B) = 3t$

$\therefore$ $A = 3$ (coeff of t) $-4A + B = 0$ (const. term)

Hence $B = 12$. So $y_p = 3t + 12$. Thus

$$y = y_c + y_p = (C_1 + C_2 t)(e^t)^{1/2} + 3t + 12$$
$$= (C_1 + C_2 \ln x).x^{1/2} + 3\ln x + 12.$$

2. (a) We know that $\mathcal{L}\{e^{at}\}(s) = 1/(s-a)$. So

$$\mathcal{L}\{\sin t\}(s) = \mathcal{L}\left\{\frac{e^{it} - e^{-it}}{2i}\right\}(s) = \frac{1}{2i}\left[\mathcal{L}\{e^{it}\}(s) - \mathcal{L}\{e^{-it}\}(s)\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-i} - \frac{1}{s+i}\right] = \frac{1}{2i} \cdot \frac{2i}{(s-i)(s+i)} = \frac{1}{s^2+1}$$

(b) $\mathcal{L}\{t^2 . f(t)\}(s) = (-1)^2 . \frac{d^2}{ds^2}\left[\mathcal{L}\{f(t)\}(s)\right]$. So

$$\mathcal{L}\{t^2 . \sin t\}(s) = (-1)^2 \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= -2 . \frac{d}{ds}\left[\frac{s}{(s^2+1)^2}\right] = -2 . \frac{1.(s^2+1)^2 - s.2.(s^2+1).2s}{(s^2+1)^4}$$

$$= -2 . \frac{(s^2+1)}{(s^2+1)^4}\left[(s^2+1) - 4s^2\right]$$

$$= -2 . \frac{1}{(s^2+1)^3} . (1 - 3s^2) = \frac{2(3s^2-1)}{(s^2+1)^3}$$

3(a) We have $y'(t) + 2y(t) = 6 \cdot e^t$ & $y(0) = 5$

So $\mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} = 6 \cdot \mathcal{L}\{e^t\}$

$\therefore\ s\,\mathcal{L}\{y\} - y(0) + 2\,\mathcal{L}\{y\} = 6 \cdot \mathcal{L}\{e^t\}$

$$\therefore\quad (s+2)\,\mathcal{L}\{y\} = 6 \cdot \frac{1}{s-1} + y(0) = \frac{6}{s-1} + 5$$

$$\therefore\quad (s+2)\,\mathcal{L}\{y\} = \frac{5s+1}{s-1}$$

$$\text{So}\quad \mathcal{L}\{y\} = \frac{5s+1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$$\therefore\quad 5s+1 = A(s+2) + B(s-1)$$

Putting $s=1$ gives us $5(1)+1 = A(1+2) \Rightarrow A = 2$

Putting $s=-2$ gives us $5(-2)+1 = B(-2-1) \Rightarrow B = 3$

$$\therefore\quad \mathcal{L}\{y\} = \frac{2}{s-1} + \frac{3}{s+2} \Rightarrow y(t) = 2e^t + 3e^{-2t}.$$

(b) $y''(t) + 4y'(t) + 5y(t) = 0$ & $y(0) = 3$, $y'(0) = -8$

So $\mathcal{L}\{y''\} + 4\,\mathcal{L}\{y'\} + 5\,\mathcal{L}\{y\} = 0$

$\therefore\ \left[s^2\mathcal{L}\{y\} - s.y(0) - y'(0)\right] + 4\left[s\,\mathcal{L}\{y\} - y(0)\right] + 5\,\mathcal{L}\{y\} = 0$

$\therefore\ (s^2+4s+5)\,\mathcal{L}\{y\} = s.y(0) + y'(0) + 4.y(0) = 3s+4$

$$\therefore\quad \mathcal{L}\{y\} = \frac{3s+4}{s^2+4s+5} = \frac{3(s+2)}{(s+2)^2+1} + \frac{-2}{(s+2)^2+1}$$

$$\therefore\quad y(t) = 3e^{-2t}\cos t - 2.e^{-2t}\sin t.$$

4(a) x_0 is a singular point of $y'' + P_1(x)y' + P_2(x)y = 0$ $(**)$ if at least one of $P_1(x)$ & $P_2(x)$ is not analytic at x_0.

x_0 is a regular singular point of $(**)$ if x_0 is a singular point of $(**)$ and $(x-x_0)P_1(x)$ & $(x-x_0)^2P_2(x)$ are both analytic at x_0.

(b) (i) The Cauchy-Euler ODE that corresponds to the ODE $2x^2y'' - xy' + xy = 0$ is $2x^2y'' - xy + 0.y = 0$.

4(b) (i) So the indicial equation will be $2r(r-1) - r + 0 = 0$

$\therefore\ 2r^2 - 3r = 0$. So $r(2r-3) = 0$. Hence

$r_1 = 3/2$ & $r_2 = 0$. Since $r_1 - r_2$ is not an integer, there will be two linearly indep. series solutions of the forms

$$y_1(x) = x^{3/2} \cdot \sum_{n=0}^{\infty} a_n . x^n \quad \& \quad y_2(x) = x^0 \cdot \sum_{n=0}^{\infty} b_n x^n \quad \text{with } a_0 = 1 = b_0.$$

(ii) The Cauchy-Euler ODE that corresponds to the ODE

$4x^2y'' + (x+1)y = 0$ is $4x^2y'' + 0.xy' + 1.y = 0$

So the indicial equation will be $4r(r-1) + 1 = 0$

$\therefore\ 4r^2 - 4r + 1 = 0$. So $(2r-1)(2r-1) = 0$. Hence

$r_1 = 1/2 = r_2$. Since $r_1 = r_2$ there will be two linearly indep. series solutions of the forms

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \& \quad y_2(x) = y_1(x) . \ln x + x^{\frac{1}{2}} \sum_{n=1}^{\infty} b_n . x^n$$

with $a_0 = 1$. Here it is possible that all the b_n's could be 0.

5. Let $y = \sum_{n=0}^{\infty} a_n . x^n$. Then $y' = \sum_{n=0}^{\infty} n . a_n . x^{n-1} = \sum_{n=1}^{\infty} n . a_n . x^{n-1}$

and $y'' = \sum_{n=0}^{\infty} n(n-1) a_n . x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$. So

$y'' - xy' + 2y = 0$ becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} . x^n - x . \sum_{n=1}^{\infty} n . a_n . x^{n-1} + 2 . \sum_{n=0}^{\infty} a_n . x^n = 0$$

$$\therefore \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n . a_n x^n + \sum_{n=0}^{\infty} 2a_n . x^n = 0$$

$$\therefore (2a_2 + 2a_0) + \sum_{n=1}^{\infty} \{(n+2)(n+1) a_{n+2} - (n-2) a_n\} . x^n = 0$$

$\therefore\ a_2 = -a_0$ & $(n+2)(n+1) a_{n+2} = (n-2) a_n$ for $n \geq 1$

$\therefore\ a_{n+2} = \dfrac{(n-2) a_n}{(n+2)(n+1)}$ for $n \geq 1$. So $a_n = \dfrac{(n-4) a_{n-2}}{n(n-1)}$ for $n \geq 3$.

Now $y(0) = \sum_{n=0}^{\infty} a_n . 0^n = a_0$ & $y'(0) = \sum_{n=1}^{\infty} n . a_n . 0^{n-1} = a_1$. Hence

$a_0 = 1$, $a_1 = 3$, $a_2 = -1$, $a_3 = \dfrac{(3-4)}{3(2)} . a_1 = -\dfrac{1}{2}$, $a_4 = 0$, and

$a_5 = \dfrac{5-4}{5(4)} . a_3 = \dfrac{-1}{40}$. $\therefore\ y(x) = 1 + 3x - x^2 - \dfrac{x^3}{2} - \dfrac{x^5}{40} - \ldots$

6 (a) Let $y = x^r \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ where $a_0 \neq 0$. Then $y = \sum_{n=0}^{\infty} a_n \cdot x^{n+r}$

So $y' = \sum_{n=0}^{\infty} (n+r) a_n \cdot x^{n+r-1}$ & $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n \cdot x^{n+r-2}$

So our ODE becomes

$$x^2 \cdot \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n \cdot x^{n+r-2} - x \cdot \sum_{n=0}^{\infty} a_n \cdot x^{n+r} + \frac{1}{4} \cdot \sum_{n=0}^{\infty} a_n \cdot x^{n+r} = 0$$

$$\therefore \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n \cdot x^{n+r} + \frac{1}{4} \sum_{n=0}^{\infty} a_n \cdot x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} \cdot x^{n+r} = 0$$

$$\therefore \left[r(r-1) + \frac{1}{4}\right] a_0 \cdot x^r + \sum_{n=1}^{\infty} \left\{\left[(n+r)(n+r-1) + \frac{1}{4}\right] a_n - a_{n-1}\right\} x^{n+r} = 0$$

So the indicial equation is $r(r-1) + \frac{1}{4} = 0$ and the recurrence relation is $[(n+r)(n+r-1) + 1/4] a_n = a_{n-1}$, for $n \geq 1$

(b) $r^2 - r + 1/4 = 0 \Rightarrow (r - 1/2)(r - 1/2) = 0 \Rightarrow r_1 = 1/2$ & $r_2 = 1/2$

So $$a_n = \frac{a_{n-1}}{(n+\frac{1}{2})(n-\frac{1}{2}) + 1/4} = \frac{a_{n-1}}{(n^2 - \frac{1}{4}) + \frac{1}{4}} = \frac{a_{n-1}}{n^2}$$

$$\therefore a_1 = \frac{a_0}{1^2} = a_0$$

$$a_2 = \frac{a_1}{(2)^2} = \frac{a_0}{4}$$

$$a_3 = \frac{a_2}{(3)^2} = \frac{a_0/4}{9} = \frac{a_0}{36}$$

$$\therefore y(x) = a_0 \cdot x^{1/2} \cdot \left[1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \cdots\right]$$

will be one solution. To get a specific solution just take $a_0 = 1$.