MAP 2302 - Differential Equations　　Florida Int'l Univ.

TEST #1 - Fall 2008　　TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. No CALCULATORS or formula sheets are allowed.

(15) 1. (a) Find the solution of the initial value problem

$$\frac{dy}{dx} - 2y = (2x-3)e^{2x} \quad \text{with} \quad y(0)=1.$$

(b) Define exactly when the function $f(x,y)$ is homogeneous.

(15) 2. Find the general solution of the ODE

$$(3y^2 - x^2)\,dx - 2xy\,dy = 0.$$

(15) 3. Find the general solution of the ODE

$$\frac{dy}{dx} - \frac{2}{x}y = 6y^2.$$

(15) 4. A body of temperature 30°C is placed in a room which is kept at a constant temperature of 10°C. After 2 hours, the temperature of the body is 20°C. What was its temperature after 1 hour?

(20) 5. A ball of mass 3kg is thrown vertically upwards from the ground with velocity $20\,ms^{-1}$. If the air resistance is λv where $\lambda = 3\,kgs^{-1}$ and g is $10\,ms^{-2}$, find

(a) the time it takes for the ball to reach its greatest height, &

(b) the greatest height the ball reaches.

(20) 6 (a) Define what is the total differential of $F(x,y)$ & what is an integrating factor of the non-exact ODE $Mdx + Ndy = 0$.

(b) Find the general solution of $\left(x^3 - \frac{y}{x}\right)dx + (1+xy)\,dy = 0.$

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Solutions to Test #1 — Fall 2008

1(a) $dy/dx - 2y = (2x-3)e^{2x}$. This is a linear 1st order ODE, so integrating factor $= e^{\int P(x)dx} = e^{\int -2dx} = e^{-2x}$.

$\therefore \quad e^{-2x}.(dy/dx) - 2y\,e^{-2x} = (2x-3)e^{2x}.e^{-2x} = 2x-3$

$\therefore \quad \frac{d}{dx}(y.e^{-2x}) = 2x-3. \quad \therefore\ y.e^{-2x} = x^2-3x+C$

But $y(0)=1$, so $1.e^{-0} = 0^2-3(0)+C \Rightarrow C=1$

$\therefore \quad ye^{-2x} = x^2-3x+1$. Hence $y = (x^2-3x+1).e^{2x}$

(b) The function $f(x,y)$ is homogeneous if we can find a real number k such that $f(tx,ty) = t^k.f(x,y)$.

2. $(3y^2-x^2)\,dx - 2xy = 0$. So $2xy\,dy = (3y^2-x^2)\,dx$

$\therefore \quad dy/dx = (3y^2-x^2)/(2xy) = \frac{3}{2}\frac{y}{x} - \frac{1}{2}\frac{x}{y}. \quad (*)$

This is a homogeneous first order ODE, so put

$v = y/x$. Then $y = xv$ and $dy/dx = v + x\,dv/dx$

So $(*)$ becomes

$$v + x\frac{dv}{dx} = \frac{3}{2}v - \frac{1}{2}.\frac{1}{v}. \quad \therefore\ x\frac{dv}{dx} = \frac{1}{2}v - \frac{1}{2}.\frac{1}{v}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1}{2}\left(\frac{v^2-1}{v}\right) \qquad \therefore\ \frac{2v}{v^2-1}\,dv = \frac{dx}{x}$$

$$\therefore\ \ln(v^2-1) = \ln x + C. \quad \therefore\ e^{\ln(v^2-1)} = e^{\ln x}.e^C$$

So $v^2-1 = x.A$ where $A = e^C$.

$\therefore \quad v^2 = 1+Ax$. But $v = y/x$, so

$y^2/x^2 = 1+Ax$

$\therefore \quad y^2 = x^2(1+Ax)$ is the general solution of the given ODE.

3. $dy/dx - (2/x)y = 6y^2$. This is a Bernoulli ODE.

So put $v = y^{1-2} = y^{-1}$. Then $\frac{dv}{dx} = \frac{dv}{dy}.\frac{dy}{dx} = -y^{-2}\frac{dy}{dx}$.

3. Now multiply both sides of the given ODE by $(1-2)y^{-2}$.
Then $-y^{-2}\frac{dy}{dx} - \left(-\frac{2}{x}\right).y.y^{-2} = -6.y^2.y^{-2}$

$\therefore\ -y^{-2}\frac{dy}{dx} + \frac{2}{x}.y^{-1} = -6$

$\therefore\ dv/dx + (2/x)v = -6$, a linear ODE.

Integrating factor $= e^{\int(2/x)dx} = e^{2\ln x} = e^{\ln(x^2)} = x^2$.

$\therefore\ x^2.(dv/dx) + 2xv = -6x^2$

$\therefore\ \frac{d}{dx}(x^2.v) = -6x^2$. So $x^2v = -2x^3 + C$

But $v = y^{-1}$, so $x^2y^{-1} = C - 2x^3$. $\therefore\ y = x^2/(C-2x^3)$.

4. We know from Newton's Law of cooling that $\frac{dT}{dt} = -k(T-T_R)$.
Also we are given that $T(0) = 30°C$ and $T(2) = 20°C$

So $\frac{dT}{dt} = -k.(T-10)$. $\therefore\ \frac{dT}{T-10} = -kdt$

$\therefore\ \ln(T-10) = -kt + C$. $\therefore\ e^{\ln(T-10)} = e^{-kt}.e^C$

$\therefore\ T-10 = A.e^{-kt}$ where $A = e^C$.

Since $T(0) = 30$, $30 - 10 = A.e^{-0} \Rightarrow A = 20$

$\therefore\ T - 10 = 20e^{-kt}$. Also since $T(2) = 20$,

$20 - 10 = 20.e^{-2k}$. So $e^{2k} = 20/10 = 2$.

$\therefore\ 2k = \ln 2$. So $k = (\ln 2)/2$. Hence

$T - 10 = 20.e^{-t(\ln 2)/2}$

So $T(1) = 10 + 20.e^{-1.(\ln 2)/2} = 10 + 20.e^{-\frac{1}{2}\ln 2}$

$= 10 + 20.e^{\ln(2^{-1/2})} = 10 + 20.2^{-1/2}$

$= 10 + 20.\frac{1}{\sqrt{2}} = 10 + 20.\frac{\sqrt{2}}{2} = 10(1+\sqrt{2})°C$.

5. From Newton's 2nd Law of Motion, we get $m\frac{dv}{dt} = -mg - \lambda v$

So $3\frac{dv}{dt} = -3(10) - 3v$. $\therefore\ \frac{dv}{dt} = -(10+v)$

So $\frac{dv}{10+v} = -dt$. $\therefore\ \ln(10+v) = -t + C_1$.

5. Hence $e^{\ln(10+v)} = e^{-t}.e^{c_1}$. So $10+v = A.e^{-t}$ where $A = e^{c_1}$. But $v(0) = 20$. So $10+20 = A.e^{-0}$. Thus $A = 30$ and so $v(t) = 30e^{-t} - 10$.

(a) The ball will reach its greatest height when $v(t)=0$, i.e., when $0 = 30e^{-t} - 10$. So $30e^{-t} = 10$ and hence $e^t = 30/10 = 3$. So $t = \ln(3)$ seconds.

(b) $v(t) = dx/dt = 30e^{-t} - 10$. $\therefore x(t) = -30e^{-t} - 10t + C_2$. But $x(0) = 0$. So $0 = -30.e^{-0} - 10(0) + C_2$. $\therefore C_2 = 30$

So $x(t) = 30 - 30e^{-t} - 10t$. Greatest height $\doteq x(\ln 3) = 30 - 30e^{-\ln 3} - 10\ln 3 = 30 - 30e^{\ln(1/3)} - 10\ln 3$

$= (30 - \frac{30}{3}) - 10\ln 3 = 10(2 - \ln 3)$ metres.

6(a) The total differential of $F(x,y)$ is defined by $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$. An integrating factor of $Mdx + Ndy = 0$ is any function $\mu = \mu(x,y)$ such that $(\mu M)dx + (\mu N)dy = 0$ is an exact ODE.

(b) Let $M = x^3 - y/x$ and $N = 1 + xy$. Then $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ $= (0 - 1/x) - (0 + y) = -(1/x + y) = -(1/x)(1+xy)$.

So the given ODE is not exact. But

$$\frac{1}{N}\cdot\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -(1/x)(1+xy)/(1+xy) = -1/x.$$

So an integrating factor $\mu = e^{\int -(1/x)dx} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x}$.

$\therefore (\mu M)dx + (\mu N)dy = \underbrace{(x^2 - y/x^2)}_{M_2}dx + \underbrace{(1/x + y)}_{N_2}dy = 0.$

$\frac{\partial F}{\partial x} = M_2 = x^2 - y/x^2$ and $\frac{\partial F}{\partial y} = N_2 = 1/x + y$.

$\therefore F = \int\left(x^2 - \frac{y}{x^2}\right)\partial x = \frac{x^3}{3} + \frac{y}{x} + \varphi(y)$.

So $\frac{\partial F}{\partial y} = 0 + \frac{1}{x} + \frac{d\varphi}{dy}$. But $\frac{\partial F}{\partial y} = \frac{1}{x} + y$

Hence $\frac{d\varphi}{dy} = y \Rightarrow \varphi(y) = \frac{y^2}{2} + C_1$. So $F = \frac{x^3}{3} + \frac{y}{x} + \frac{y^2}{2} + C_1$

Since $dF = M_2\,dx + N_2\,dy = 0$, $F(x,y) = C_2$. Hence

$\frac{x^3}{3} + \frac{y}{x} + \frac{y^2}{2} + C_1 = C_2$. $\therefore \frac{x^3}{3} + \frac{y}{x} + \frac{y^2}{2} = C$ where $C = C_2 - C_1$.