MAP 2302 – Differential Equations — Florida Int'l Univ.

TEST # 2 – Fall 2008 — TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. NO CALCULATORS OR FORMULA SHEETS ARE ALLOWED.

(15) 1. Find the solution of each of the following homog. ODEs

(a) $y'' + 2y' + y = 0$ with $y(0) = 2$ and $y'(0) = 1$

(b) $y'' + 16y = 0$ with $y(0) = 3$ and $y'(0) = -8$

(20) 2. Find the general solution of each of the following ODEs

(a) $y'' - y = 12e^{x}$ (b) $y'' - y' - 2y = -4x^2$.

(20) 3. Find y_c and give the <u>minimal form</u> of the y_p that one should try for each of the following ODEs

(a) $(D^2 + 2D + 2)^2 y = 6x \cdot e^{-x} \cdot \sin x$

(b) $(D^2+1)^2 \cdot (D^2-4) y = 4\cos^2 x + 5\sin x$

(15) 4(a) Define what is the Wronskian of $\{f_1, f_2, f_3\}$.

(b) Find a particular solution of the ODE $y'' + y = \csc^2(x)$.

(15) 5(a) Define what it means for $\{f_1, f_2, f_3\}$ to be linearly dependent.

(b) Given that $f(x) = x$ is a solution of $(x^2+3)y'' - 4x \cdot y' + 4y = 0$, find a second linearly independent solution.

(15) 6. A body of mass 3kg is attached to a linear spring of natural length 5m and the other end of the spring is attached to the ceiling. The spring constant k is $15\ Nm^{-1}$ and the air resistance is λv where $\lambda = 6\ Nsm^{-1}$. If the body is released from rest with the spring un-extended at time $t=0$, find its position at all subsequent times. [Use $g = 10\ ms^{-2}$]

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Solutions to Test # 2	Fall 2008

1 (a) $y'' + 2y' + y = 0$	$y(0) = 2$ & $y'(0) = 1$

$(D^2 + 2D + 1)y = 0 \Rightarrow (D+1)^2 y = 0 \Rightarrow D = -1$ (twice)

$\therefore\ y = (Ax + B).e^{-x}$

$\therefore\ y' = (A.1 + 0).e^{-x} + (Ax+B).(-e^{-x}) = (A - B - Ax)e^{-x}$

$y(0) = 2 \Rightarrow (A.0 + B).e^{-0} = 2 \Rightarrow B = 2$

$y'(0) = 1 \Rightarrow [A - B - A.0]e^{-0} = 1 \Rightarrow B = 1+2 = 3$

$\therefore\ y(x) = (3x+2)e^{-x}$.

(b) $y'' + 16y = 0$,	$y(0) = 3$ & $y'(0) = -8$

$(D^2 + 16)y = 0 \Rightarrow (D - 4i)(D + 4i)y = 0$

$\therefore\ y = A\cos(4x) + B\sin 4x$

$y' = -4A\sin 4x + 4B\cos 4x$

$y(0) = 3 \Rightarrow A.1 + B.0 = 3 \Rightarrow A = 3$

$y'(0) = -8 \Rightarrow -4A.0 + 4B = -8 \Rightarrow B = -2$

$\therefore\ y(x) = 3\cos(4x) - 2\sin(4x)$.

2 (a) $y'' - y = 12e^{x}$ (**). Homog. Eq. $y'' - y = 0$

$\therefore (D^2 - 1)y = 0 \Rightarrow (D-1)(D+1)y = 0 \Rightarrow D = 1$ or -1.

$\therefore\ y_c = C_1 e^{x} + C_2 e^{-x}$

Since 1 is a root of the aux. eq. & 12 is a polynomial of degree 0, try $y_p = x^1.(A.e^{x})$

$\therefore\ y_p' = 1.Ae^{x} + x.A.e^{x} = A(x+1).e^{x}$

$y_p'' = A.(1+0).e^{x} + A(x+1).e^{x} = (Ax + 2A)e^{x}$

So (**) becomes $(Ax + 2A)e^{x} - Axe^{x} = 12e^{x}$

$\therefore\ 2Ae^{x} = 12e^{x}$.	$\therefore\ A = 12/2 = 6$.

$\therefore\ y_p = Ax.e^{x} = 6xe^{x}$.

$\therefore\ y(x) = y_c(x) + y_p(x) = C_1 e^{x} + C_2 e^{-x} + 6xe^{x}$.

2(b) $y'' - y' - 2y = -4x^2$ (**). Homog. Eq. $y'' - y' - 2y = 0$

$(D^2 - D - 2)y = 0 \Rightarrow (D+1)(D-2)y = 0 \Rightarrow D = -1$ or 2

$\therefore\ y_c = C_1 e^{-x} + C_2 e^{2x}$. Since $-4x^2$ is a polynomial of degree 2 & 0 is not a root of the aux. eq., try $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B, \quad y_p'' = 2A$

$\therefore$ (**) becomes

$(2A) - (2Ax + B) - 2(Ax^2 + Bx + C) = -4x^2$

$\therefore\ -2Ax^2 - (2B + 2A)x + (2A - B - 2C) = -4x^2$

$\therefore\ -2A = -4 \Rightarrow A = 2$

$2B + 2A = 0 \Rightarrow B = -A = -2$

$2A - B - 2C = 0 \Rightarrow 2C = -B + 2A \Rightarrow C = 3.$

$\therefore\ y_p = 2x^2 - 2x + 3$. $\therefore\ y = y_c + y_p = C_1 e^{-x} + C_2 e^{2x} + 2x^2 - 2x + 3.$

3(a) $(D^2 + 2D + 2)^2 y = 6x \cdot e^{-x} \cdot \sin x$

$(D^2 + 2D + 2)^2 y = 0 \Rightarrow D = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$ (twice). $\therefore\ y_c = (C_1 x + C_2)e^{-x}\cos x + (C_3 x + C_4)e^{-x}\sin x$

Since $6x$ is a polynomial of degree 1 & $-1 \pm i$ are roots of the aux. eq. of multiplicity 2, the minimal y_p that we should try will be

$$y_p = x^2 \cdot [(A_1 x + A_2)e^{-x}\cos x + (B_1 x + B_2)e^{-x}\sin x].$$

(b) $(D^2+1)^2(D^2-4)y = 4\cos^2 x + 5\sin x = \frac{4}{2}(1 + \cos 2x) + 5\sin x$

$= 2 + 2\cos(2x) + 5\sin x$. $(D^2+1)^2(D-2)(D+2)y = 0$

$\Rightarrow D = \pm i$ (twice), -2, or 2.

$\therefore\ y_c = (C_1 x + C_2)\cos x + (C_3 x + C_4)\sin x + C_5 e^{-2x} + C_6 e^{2x}$.

Since $\pm i$ are roots of the aux. eq. of multiplicity 2, the minimal y_p that we should try will be

$$y_p = A_0 + A_1\cos(2x) + B_1\sin(2x) + x^2[A_2\cos x + B_2\sin x]$$

4(a) $W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$

(b) $y'' + y = \csc^2(x)$ $\quad (D^2+1)y = 0 \Rightarrow D = \pm i$

So we can take $y_1 = \cos x$ and $y_2 = \sin x$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$\therefore \; v_1' = \dfrac{\begin{vmatrix} 0 & y_2 \\ F(x)/a_0(x) & y_2' \end{vmatrix}}{W(y_1, y_2)} = \dfrac{\begin{vmatrix} 0 & \sin x \\ \csc^2 x & \cos x \end{vmatrix}}{1} = \dfrac{-\csc^2 x . \sin x}{1}$

$= -\cancel{\csc x} . \dfrac{1}{\sin x} . \sin x = -\csc(x)$

$\therefore \; v_1 = \int -\csc(x)\, dx = --\ln(\csc x + \cot x)$

$v_2' = \dfrac{\begin{vmatrix} y_1 & 0 \\ y_2' & F(x)/a_0(x) \end{vmatrix}}{W(y_1, y_2)} = \dfrac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \csc^2 x \end{vmatrix}}{1}$

$= \cos x . \csc^2 x = \cos x . \dfrac{1}{\sin x} . \csc x = \cot x \csc x$

$\therefore \; v_2 = \int \cot x \csc\, dx = -\csc x .$

$\therefore \; y_p = y_1 v_1 + y_2 v_2 = \cos x . \ln(\csc x + \cot x) - \csc x . \sin x$

$= \cos x . \ln(\csc x + \cot x) - 1 .$

5(a) $\{f_1, f_2, f_3\}$ is linearly dependent if we can find c_1, c_2, c_3 with at least one them being non-zero such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) \equiv 0$.

(b) $(x^2+3)y'' - 4xy' + 4y = 0$, one solution is $f(x) = x$.

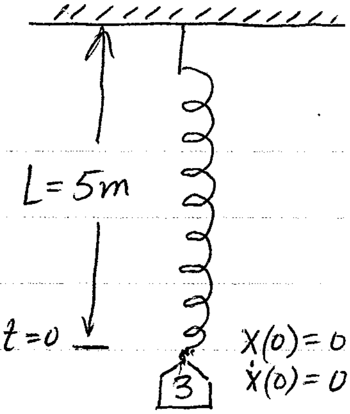
$y_2(x) = f(x).v = x . \int \dfrac{e^{-\int a_1(x)/a_0(x)\, dx}}{[f(x)]^2}\, dx$

$= x . \int \frac{1}{x^2} . \left(e^{-\int \frac{-4x}{x^2+3} dx}\right) dx = x . \int \dfrac{e^{2\ln(x^2+3)}}{x^2} dx$

$= x . \int \dfrac{(x^2+3)^2}{x^2} dx = x . \int \dfrac{x^4 + 6x^2 + 9}{x^2} dx$

$= x . \int \left(x^2 + 6 + \frac{9}{x^2}\right) dx = x . \left(\frac{x^3}{3} + 6x - \frac{9}{x}\right) = \frac{x^4}{3} + 6x^2 - 9 .$

6. Let x = the amount the spring is extended. Then $x(0) = 0$ & $\dot{x}(0) = 0$. Also



$$m\ddot{x} = -kx - \lambda\dot{x} + mg$$

$$\therefore \quad 3\ddot{x} = -15x - 6\dot{x} + 3(10)$$

$$\therefore \quad \ddot{x} + 2\dot{x} + 5x = 10$$

$$(D^2 + 2D + 5)x = 0$$

$$\therefore \quad D = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$\therefore \quad x_c(t) = e^{-t}[A\cos(2t) + B\sin(2t)]$$

Try $x_p(t) = C$. Then $\dot{x}_p = 0$ & $\ddot{x}_p = 0$

So $\ddot{x} + 2\dot{x} + 5x = 10$ becomes $0 + 0 + 5C = 10$

$\therefore \quad C = 2. \quad \therefore \quad x_p(t) = 2.$

$$\therefore \quad x(t) = x_c(t) + x_p(t) = 2 + e^{-t}(A\cos(2t) + B\sin(2t))$$

and
$$\dot{x}(t) = 0 + e^{-t}(-2A\sin(2t) + 2B\cos(2t)) - e^{-t}[A\cos(2t) + B\sin(2t)]$$
$$= e^{-t}.[(2B - A)\cos 2t - (2A + B)\sin(2t)]$$

$x(0) = 0 \Rightarrow 0 = 2 + e^{-0}(A.1 + B.0) \Rightarrow A = -2$

$\dot{x}(0) = 0 \Rightarrow 0 = (2B - A).1 - (2A + B).0 \Rightarrow B = -1$

$$\therefore \quad x(t) = 2 - e^{-t}[2\cos(2t) + \sin(2t)].$$