MAP 2302 - Differential Equations   Florida Int'l Univ.

Test #3 - Fall 2008   TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(15) 1. Find the general solution of the ODE $x^2y'' + \frac{y}{4} = 9x^2$ by transforming it into a linear non-homogeneous constant coefficient ODE in y and t.

(15) 2(a) Define what it means for x_0 to be a singular point and for x_0 to be a regular singular point of the ODE $y'' + P_1(x)y' + P_2(x)y = 0$.

(b) Find the general solution of the ODE $x^2y'' - xy' + 2y = 0$.

(15) 3. Starting with $\mathcal{L}\{e^{at}\}(s) = 1/(s-a)$, use the Properties of the Laplace transform to find

(a) $\mathcal{L}\{\cos t\}(s)$ (b) $\mathcal{L}\{t^2 \cos t\}(s)$.

(20) 4. Solve the following ODEs by using the Laplace Transform

(a) $y'(t) + 2y(t) = 4e^{-2t}$ with $y(0) = 3$.

(b) $y''(t) - 2y'(t) - 3y(t) = 0$ with $y(0) = 2$ & $y'(0) = -6$.

(20) 5. For each of the following ODEs, find the indicial equation and the form of two linearly independent Frobenius series solution about $x_0 = 0$.

(a) $2x^2y'' + xy' + (x-1)y = 0$ (b) $x^2y'' + \left(x - \frac{15}{4}\right)y = 0$

(15) 6. Find the first 4 non-zero terms of the power series solution of the ODE $y'' + x^2y = 0$ with the initial conditions $y(0) = 3$ & $y'(0) = 5$.

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Solutions to Test #3 Fall 2008

1. We have $x^2y'' + y/4 = 9x^2$. Put $x = e^t$ and $D = \frac{d}{dt}$. Then $x^2y'' = D(D-1)y$ & $xy' = Dy$.

So our ODE becomes $D(D-1)y + y/4 = 9.(e^t)^2$

$\therefore (D^2 - D + 1/4)y = 9e^{2t}$. $(D-1/2)^2 y = 9.e^{2t}$

$\therefore y_c = c_1.e^{t/2} + c_2.t e^{t/2} = c_1.x^{1/2} + c_2.\ln(x).x^{1/2}$

Let $y_p = A.e^{2t}$. Then $\dot{y}_p = 2Ae^{2t}$ & $\ddot{y}_p = 4Ae^{2t}$

So our ODE becomes $4Ae^{2t} - 2Ae^{2t} + \frac{1}{4}Ae^{2t} = 9e^{2t}$

$\therefore (2\frac{1}{4})A = 9$. $\therefore A = 4$. So $y_p = 4e^{2t} = 4x^2$

$\therefore y = y_c + y_p = x_1^{1/2}(c_1 + c_2 \ln x) + 4x^2$.

2(a) x_0 is a singular point of $y'' + P_1(x)y' + P_2(x)y = 0$ if at least one of the two functions $P_1(x)$ & $P_2(x)$ is not analytic at x_0. x_0 is a regular singular point if x_0 is a singular point and both $(x-x_0)P_1(x)$ and $(x-x_0)^2P_2(x)$ are analytic at x_0.

(b) We have $x^2y'' - xy' + 2y = 0$. Put $x = e^t$ and $D = \frac{d}{dt}$. Then $xy' = Dy$ and $x^2y'' = D(D-1)y$.

So our ODE becomes $D(D-1)y - Dy + 2y = 0$

$\therefore (D^2 - 2D + 2)y = 0$

$\therefore D = (2 \pm \sqrt{4-8})/2 = (2 \pm 2i)/2 = 1 \pm i$

$\therefore y(t) = c_1e^t.\cos t + c_2e^t \sin t$

$\therefore y(x) = c_1.x.\cos(\ln x) + c_2.x.\sin(\ln x)$

3(a) $\mathcal{L}\{\cos t\}(s) = \mathcal{L}\{(e^{it} + e^{-it})/2\}(s) = 1/2\left[\mathcal{L}\{e^{it}\}(s) + \mathcal{L}\{e^{-it}\}(s)\right]$

$$= \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) = \frac{1}{2}\frac{(s+i)+(s-i)}{s^2-i^2} = \frac{s}{s^2+1}.$$

3(b) $\mathcal{L}\{t^2\cos t\}(s) = (-1)^2 \cdot \frac{d^2}{ds^2}\left[\mathcal{L}\{\cos t\}(s)\right] = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$

$= \frac{d}{ds}\left[1 \cdot \frac{1}{s^2+1} + s \cdot \frac{(-1).(2s)}{(s^2+1)^2}\right] = \frac{d}{ds}\left[\frac{1}{s^2+1} - \frac{2s^2}{(s^2+1)^2}\right]$

$= \frac{(-1).(2s)}{(s^2+1)^2} - \frac{4s}{(s^2+1)^2} - 2s^2 \cdot \frac{(-2)(2s)}{(s^2+1)^3}$

$= \frac{-6s \cdot (s^2+1)}{(s^2+1)^3} + \frac{8 \cdot s^3}{(s^2+1)^3} = \frac{2s(s^2-3)}{(s^2+1)^3}$.

4(a) $y'(t) + 2y(t) = 4e^{-2t}$ and $y(0) = 3$

$\therefore\ \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 4\mathcal{L}\{e^{-2t}\}$

$\therefore\ s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\} = 4/(s+2)$

$\therefore\ (s+2)\mathcal{L}\{y\} = 4/(s+2) + y(0) = 4/(s+2) + 3$

$\therefore\ \mathcal{L}\{y\} = \frac{4}{(s+2)^2} + \frac{3}{s+2}$

$\therefore\ y(t) = 4t \cdot e^{-2t} + 3e^{-2t}$

(b) $y''(t) - 2y'(t) - 3y(t) = 0$ and $y(0) = 2$ & $y'(0) = -6$

$\therefore\ \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 0$

$\therefore\ s^2\mathcal{L}\{y\} - s.y(0) - y'(0) - 2[s\mathcal{L}\{y\} - y(0)] - 3\mathcal{L}\{y\} = 0$

$\therefore\ (s^2 - 2s - 3)\mathcal{L}\{y\} = s\,y(0) + y'(0) - 2y(0)$

$\therefore\ (s+1)(s-3)\mathcal{L}\{y\} = 2s - 6 - 2(2) = 2s - 10$

$\therefore\ \mathcal{L}\{y\} = \frac{2s-2}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} = \frac{A(s+1) + B(s-3)}{(s-3)(s+1)}$

$\therefore\ A(s+1) + B(s-3) = 2s - 10$

Putting $s = 3$, gives $A(3+1) = 2(3) - 10 \Rightarrow A = -1$

Putting $s = -1$, gives $B(-1-3) = 2(-1) - 10 \Rightarrow B = 3$

$\therefore\ \mathcal{L}\{y\} = 3/(s+1) - 1/(s-3)$

$\therefore\ y(t) = 3e^{-t} - e^{3t}$.

5(a) $2x^2y'' + xy' + (x-1)y = 0$. So $2x^2y'' + xy' - y = 0$ will be the associated Cauchy-Euler equation.

5 (a) $\therefore\ 2r(r-1)+r-1=0$ will be the indicial equation.

$\therefore\ 2r^2-r-1=0 \quad \therefore\ (2r+1)(r-1)=0$, So

$r_1=1$ and $r_2=-1/2$. Since $r_1-r_2 \notin \mathbb{N}$, we get two linearly indep. solutions of the form

$y_1(x)=x^1.\sum_{n=0}^{\infty} a_n.x^n$ and $y_2(x)=x^{-1/2}.\sum_{n=0}^{\infty} b_n.x^n$

with $a_0=1=b_0$.

(b) $x^2y''+\left(x-\frac{15}{4}\right)y=0$. So $x^2y''-(15/4)y=0$ will be the associated Cauchy-Euler Equation. So the indicial equation will be $r(r-1)-15/4=0$

$\therefore\ r^2-r-15/4=0$, $r=(1\pm\sqrt{1+15})/2 = 5/2$ or $-3/2$

$\therefore\ r_1=5/2$ & $r_2=-3/2$. Since $r_1-r_2=4\in\mathbb{N}^+$, we get two linearly independent solutions of the form

$y_1(x)= x^{5/2}.\sum_{n=0}^{\infty} a_n.x^n$ with $a_0=1$, and

$y_2(x)= A.\,y_1(x)\ln(x) + x^{-3/2}.\sum_{n=0}^{\infty} b_n\,x^n$ with $b_0=1$.

Here A is a constant which may or may not be 0.

6. $y''+x^2y=0$. $(*)$, $y(0)=3$ & $y'(0)=5$. Let $y=\sum_{n=0}^{\infty} a_nx^n$.

Then $y'=\sum_{n=0}^{\infty} n.a_nx^{n-1} = \sum_{n=0}^{\infty}(n+1).a_{n+1}.x^n$ and

$y''=\sum_{n=0}^{\infty} n(n-1).a_n.x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$.

So $(*)$ becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2.\sum_{n=0}^{\infty}a_n.x^n = 0$$

$$\therefore\ \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2}x^n = 0$$

$$\therefore\ 2a_2.x^0+6a_3.x^1+\sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2}+a_{n-2}\}x^n=0$$

$\therefore\ 2a_2=0, 6a_3=0$ & $(n+2)(n+1)a_{n+2}+a_{n-2}=0$, $n\geq 2$

$\therefore\ a_2=0, a_3=0$ & $a_{n+2} = -a_{n-2}/(n+2)(n+1)$, $n\geq 2$.

$y(0)=3 \Rightarrow a_0=3$ & $y'(0)=5\Rightarrow a_1=5$.

$a_4=-a_0/(2+2)(2+1)=-3/12=-1/4$, $a_5=-a_1/20=-1/4$.

$\therefore\ y(x)= 3+5x+0.x^2+0.x^3-\frac{x^4}{4}-\frac{x^5}{4}+\ldots$