Notes on linear ODE's with regular singular points

Let $x_0 = 0$ be a regular singular point of the ODE

$$x^2 y'' + x.P(x).y' + Q(x)y = 0 \quad \ldots (*)$$

and r_1 & r_2 be the roots of the indicial equation of (*).
Here $P(x)$ & $Q(x)$ are analytic function of x because $x_0 = 0$ is a regular singular point of (*). So

$$P(x) = p_0 + p_1 x + p_2 x^2 + \cdots = \sum_{n=0}^{\infty} p_n x^n .$$

$$\& \quad Q(x) = q_0 + q_1 x + q_2 x^2 + \cdots = \sum_{n=0}^{\infty} q_n x^n .$$

The Cauchy-Euler ODE that is associated with (*) is

$$x^2 y'' + p_0 . x . y' + q_0 . y = 0 \quad \ldots (**)$$

and if we put $x = e^t$ & $\Delta = \frac{d}{dt}$, then (**) becomes

$$[\Delta(\Delta - 1) + p_0 \Delta + q_0] y = 0 .$$

So the auxiliary equation of (**) is

$$\Delta(\Delta - 1) + p_0 . \Delta + q_0 = 0 .$$

<u>Theorem 1</u> : The indicial equation of the ODE (*) is the same as the auxiliary equation of the associated Cauchy-Euler ODE (**), i.e, it is $r(r-1) + p_0 r + q_0 = 0$.

<u>Ex. 1</u>. Consider the ODE with a reg. sing. pt. at $x_0 = 0$,

$$x^2 y'' + x(3 - x + 2x^2) y' + (-2 - x + x^3) y = 0 . \ (*)$$

Then the associated Cauchy-Euler ODE is

$$x^2 y'' + 3x . y' - 2 . y = 0 . \quad \ldots (**)$$

The auxiliary of (**) is $\Delta(\Delta - 1) + 3\Delta - 2 = 0$,
So the indicial equation of (**) is therefore

$$r(r-1) + 3r - 2 = 0 .$$

Theorem 2 (Frobenius Theorem with real roots)
Suppose the roots of the indicial equation of $(*)$ are real and $r_1 \geq r_2$, $\mathbb{N} = \{0,1,2,3,\ldots\}$ & $\mathbb{N}^+ = \{1,2,3,\ldots\}$.

(a) If $r_1 - r_2 \notin \mathbb{N}$, then the ODE $(*)$ has two linearly independent solutions of the form
$$y_1(x) = x^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \quad \& \quad y_2(x) = x^{r_2} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n \quad \text{with } a_0 = 1 = b_0.$$

(b) If $r_1 - r_2 = 0$, then $(*)$ has two lin. indep. solutions of the form $y_1(x) = x^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ with $a_0 = 1$, & $y_2(x) = y_1(x) \cdot \ln(x) + x^{r_1} \cdot \sum_{n=1}^{\infty} b_n \cdot x^n$. Here it is possible for all the b_n's to be zero.

(c) If $r_1 - r_2 \in \mathbb{N}^+$, then $(*)$ has two lin. indep. solutions of the form $y_1(x) = x^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ with $a_0 = 1$, & $y_2(x) = A \cdot y_1(x) \cdot \ln(x) + x^{r_2} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$ with $b_0 = 1$. Here it is possible for A to be 0.

Theorem 3 (Frobenius Method with complex roots)
Suppose the roots of the indicial equation of $(*)$ are complex. Then one root $r_1 = \alpha + i\beta$ and the other root $r_2 = \alpha - i\beta$ where $\alpha \in \mathbb{R}$ & $\beta > 0$.

The ODE $(*)$ will have two linearly independent solutions of the form
$$y_1(x) = x^{\alpha} \cos(\beta \ln x) \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \quad \text{with } a_0 = 1,$$
$$\& \quad y_2(x) = x^{\alpha} \sin(\beta \ln x) \cdot \sum_{n=0}^{\infty} b_n \cdot x^n \quad \text{with } b_0 = 1.$$