Answer all 6 questions. ***No calculators, formula sheets, or cell phones are allowed.*** *An unjustified answer will receive little or no credit. So show all working and provide all reasoning.*

(15) 1 (a) Define what is an *integrating factor* of the non-exact differential equation

$$M(x,y).dx \; + \; N(x,y).dy \; = \; 0 .$$

(b) Find the general solution of the differential equation

$$\{3x^2 - \; y^{-1}.\, sec^2(x)\}.dx \quad + \quad \{sin(y) + \; y^{-2}.\, tan(x)\}.dy \; = \; 0.$$

(15) 2 (a) Define what it means for the function f(x,y) to be *homogeneous*.

(b) Find the solution of the differential equation $dy/dx \; - \; 3.y \; = \; 12.e^{-x}$ with $y(0) = 2$.

(15) 3. Find the general solution of the ODE $\{ x^2 + y^2\}.dx \; - \; \{ 2xy \}.dy \; = \; 0.$

(15) 4. Find the general solution to the differential equation $dy/dx - 2y/x = 4x^2.y^{1/2}$.

(20) 5. A ball of mass $2\ kg$ is thrown vertically upwards from sea level with velocity $15\ ms^{-1}$. If the acceleration due to gravity g is $10\ ms^{-2}$ and the air resistance is λv where $\lambda = 4\ kgs^{-1}$,

(a) Find the *time* it takes for the ball to reach its greatest height, and

(b) Find the *greatest height* above sea level that the ball reaches.

(20) 6. The population of a colony of micro-organisms satisfy the differential equation $dP/dt \; = \; P - (P^2 / 2000)$, where t is measured in hours.

(a) If $P(0) = 4000$, find the *population* after t hours.

(b) How *long* will it take for the population to decrease to 2400 ?

MAP 2302 - Differential Equation Florida Int'l Univ.

Solutions to Test #1 Fall 2013

1(a) An integrating factor of the non-exact ODE $M.dx + Ndy = 0$ is any function $\mu = \mu(x,y)$ such that $(\mu M)dx + (\mu N)dy = 0$ is an exact ODE.

(b) This is an exact ODE because $\partial M/\partial y = \partial N/\partial x$.
So we can find a function F such that $dF = Mdx + Ndy$.
Now $dF = (\partial F/\partial x)dx + (\partial F/\partial y)dy$, so

$\partial F/\partial x = M = 3x^2 - y^{-1}\sec^2 x$ & $\partial F/\partial y = N = \sin(y) - y^{-2}\tan x$.

$\therefore F = \int (3x^2 - y^{-1}\sec^2 x)\,\partial x = x^3 - y^{-1}\tan x + \varphi(y)$

$\therefore \partial F/\partial y = 0 + y^{-2}\tan(x) + \varphi'(y)$. But $\frac{\partial F}{\partial y} = \sin y - y^{-2}\tan x$.

Hence $\varphi'(y) = \sin(y)$. Thus $\varphi(y) = -\cos(y) + C_1$

So $F = x^3 - y^{-1}\tan x - \cos y + C_1$. But $dF = Mdx + Ndy = 0$,
so $F = C_2$. Hence $x^3 - y^{-1}\tan x - \cos y + C_1 = C_2$

Thus $x^3 - y^{-1}\tan(x) - \cos(y) = C$ where $C = C_2 - C_1$.

2(a) The function $f(x,y)$ is homogeneous if we can find a real number α such that $f(tx, ty) = t^\alpha f(x,y)$ for all x, y & t.

(b) $dy/dx - 3.y = 12.e^{-x}$. This is a linear first order ODE. So $I.F. = e^{\int -3dx} = e^{-3x}$. Thus

$e^{-3x}.(dy/dx) - 3.e^{-3x}.y = 12e^{-x}.e^{-3x}$

$\therefore \frac{d}{dx}(y.e^{-3x}) = 12e^{-4x}$

So $y.e^{-3x} = \int 12.e^{-4x}dx = -3e^{-4x} + C$

Hence $y = e^{3x}.(-3e^{-4x} + C) = Ce^{3x} - 3e^{-x}$

But $y(0) = 2$, so $2 = C.e^0 - 3e^0 = C - 3$

Hence $C = 5$. Thus $y = 5e^{3x} - 3e^{-x}$.

3. We have $(X^2+Y^2)dx = 2xy\,dy$. So $dy/dx = (x^2+y^2)/(2xy)$
$= \frac{1}{2}\frac{x}{y} + \frac{1}{2}\frac{y}{x}$. Thus this is a homogeneous ODE.
Let us put $y = xv$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ & $v = \frac{y}{x}$.
So $v + x\frac{dv}{dx} = \frac{1}{2}\frac{x}{y} + \frac{1}{2}\frac{y}{x} = \frac{1}{2}\frac{1}{v} + \frac{1}{2}v$.

$\therefore \quad x\frac{dv}{dx} = \frac{1}{2}\cdot\frac{1}{v} + \frac{1}{2}v - v = -\frac{1}{2}v + \frac{1}{2}\cdot\frac{1}{v}$

$\therefore \quad x\frac{dv}{dx} = -\frac{1}{2}\left(\frac{v^2-1}{v}\right)$. $\therefore \frac{2v\,dv}{v^2-1} = -\frac{dx}{x}$

Thus $\ln(v^2-1) = -\ln x + C$

$\therefore \quad v^2 - 1 = e^{-\ln x}\cdot e^C = A\cdot e^{\ln(\frac{1}{x})} = \frac{A}{x}$

$\therefore \quad v^2 = \frac{A}{x} + 1 \Rightarrow y^2/x^2 = Ax^{-1} + 1$

$\therefore \quad y^2 = x^2(Ax^{-1}+1) = Ax + x^2$.

4. $dy/dx - 2y/x = 4\cdot x^2\cdot y^{1/2}$. This is a Bernoulli ODE
with index $\alpha = 1/2$. So put $v = y^{1-\alpha} = y^{1/2}$ and
multiply both sides of the ODE by $(1-\alpha)y^{-\alpha} = (1/2)y^{-1/2}$.

$\therefore \quad \frac{1}{2}\cdot y^{-1/2}\frac{dy}{dx} - \frac{1}{2}\cdot y^{-1/2}\cdot\frac{2y}{x} = \frac{1}{2}\cdot y^{-1/2}\cdot 4x^2\cdot y^{1/2}$

$\therefore \quad \frac{1}{2}y^{-1/2}\frac{dy}{dx} - \frac{y^{1/2}}{x} = 2x^2$.

So $\quad \frac{dv}{dx} - \frac{v}{x} = 2x^2$. $\quad$ I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$.

$\therefore \quad \frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2} = 2x$. $\quad$ So $\frac{d}{dx}\left(\frac{v}{x}\right) = 2x$

$\therefore \quad \frac{v}{x} = \int 2x\,dx = x^2 + C \Rightarrow \frac{y^{1/2}}{x} = x^2 + C$

$\therefore \quad y^{1/2} = x(x^2+C)$. So $y = x^2(x^2+C)^2$.

5. From Newton's Second Law, we have $m\,dv/dt = -mg - \lambda v$.
So $2(dv/dt) = -2(10) - 4v$, $\therefore dv/dt = -(10+2v)$
$= -2(5+v)$

$\therefore \quad \frac{dv}{5+v} = -2dt$. So $\ln(v+5) = -2t + C$

$\therefore \quad v + 5 = e^{-2t}\cdot e^C = Ae^{-2t}$ where $A = e^C$.

5. But $v(0)=15$, so $15+5=Ae^0 \Rightarrow A=20$. Thus
$v+5=20e^{-2t} \Rightarrow v(t)=20e^{-2t}-5=5(4e^{-2t}-1)$.
$\therefore dx/dt = 20e^{-2t}-5$. $\therefore x(t)=-10e^{-2t}-5t+C_2$
But $x(0)=0$. So $0=-10-0+C_2 \Rightarrow C_2=10$. Thus
$x(t)=10-10e^{-2t}-5t=10(1-e^{-2t})-5t$.

(a) Ball will reach its greatest height when $v(t)=0$.
This is when $5(4e^{-2t}-1)=0 \Rightarrow 4e^{-2t}=1 \Rightarrow e^{2t}=4$
$\Rightarrow 2t=\ln 4=2\ln 2$. So $t=\ln 2$.

(b) Greatest height ball reaches will be $x(\ln 2)=$
$10(1-e^{-2\ln 2})-5\ln 2 = 10(1-\frac{1}{4})-5\ln 2 = 5(\frac{3}{2}-\ln 2)$.

6(a) We have $\frac{dP}{dt}=P-\frac{P^2}{2000}=\frac{2000P-P^2}{2000}=\frac{-P(P-2000)}{2000}$

$\therefore \frac{-2000\,dP}{P(P-2000)}=dt$. $\therefore \frac{1}{P}-\frac{1}{P-2000}=dt$

$\therefore \int\left(\frac{1}{P}-\frac{1}{P-2000}\right)dP=\int dt$. $\therefore \ln(P)-\ln(P-2000)=t+C$

$\therefore \ln\left(\frac{P}{P-2000}\right)=t+C$ $\therefore \frac{P}{P-2000}=e^t\cdot e^C=A.e^t$

But $P(0)=4000$. So $\frac{4000}{4000-2000}=A.e^0=A$

$\therefore A=2$. So $\frac{P}{P-2000}=2e^t$. $\therefore \frac{P-2000}{P}=\frac{e^{-t}}{2}$

$\therefore 1-(2000/P)=e^{-t}/2$. $\therefore \frac{2000}{P}=1-\frac{e^{-t}}{2}$.

$\therefore P=2000/\left(1-\frac{e^{-t}}{2}\right)=4000/(2-e^{-t})$.

(b) If $P=2400$, then $\frac{P}{P-2000}=2e^t$. So $\frac{2400}{2400-2000}=2e^t$

$\therefore \frac{2400}{600}=2e^t$. $\therefore 6=2e^t \Rightarrow 3=e^t$

Thus $t=\ln 3$. So the population will be 2400 in $\ln 3$ hours.

END.