MAP 2302 - Differential Equations | Florida International University
TEST #2 - Fall 2013 | TIME: 75 min.

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(20) 1. Find the solution of each of the following homogeneous differential equation.

(a) $y'' + 6y' + 9y = 0$ with $y(0) = -1$ and $y'(0) = 5$.

(b) $y'' + y' - 2y = 0$ with $y(0) = 1$ and $y'(0) = 7$.

(15) 2. Find the general solution of the linear ODE $y'' + 2y' - 2y = 5.\cos(x)$.

(15) 3. Find the general solution of the differential equation $y'' - y = 4x.e^{x}$.

(20) 4. Find y_c, and give the minimal form of the y_p that one should try, for each of the following linear non-homogeneous differential equations.

(a) $(D^2 - 4D + 5)^2\, y = 3 + 5x.e^{2x}.\cos(x)$.

(b) $D(D^2 - 1)(D^2 + 4)\, y = e^{-x} + 6.\cos^2(x)$.

(15) 5. (a) Define what it means for the functions f_1, f_2, f_3, f_4 of x to be linearly independent.

(b) Find a particular solution y_p of the ODE $y'' - 2y' + y = -(e^{x})/(x^2)$ by using the method of variation of parameters.

(15) 6. A body of mass $4\ kg$ is attached to a linear (Hooke-type) spring and suspended from a very high ceiling. The natural length of the spring is $10m$, the spring constant k is $5\ Nm^{-1}$, and the air resistance is λv where $\lambda = 4\ Nsm^{-1}$. If the spring is stretched by an amount of $10m$ and then set loose from rest, find the amount it will be extended at all subsequent times. [Use $g = 10\ ms^{-2}$.]

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Solutions to Test #2   Fall 2013

1(a) $y''+6y'+9y=0$. $\therefore (D^2+6D+9)y=0$. $\therefore (D+3)^2y=0$

So $D=-3$ (twice). So $y=(Ax+B)\cdot e^{-3x}$. Hence

$$y'=(A+0)e^{-3x}-3.(Ax+B).e^{-3x}=[(A-3B)-3Ax].e^{-3x}$$

$y(0)=-1 \Rightarrow (A.0+B).e^0=-1 \Rightarrow B=-1$

$y'(0)=5 \Rightarrow [(A-3B)-3A.0].e^0=5 \Rightarrow A=5+3B=2$.

$\therefore \quad y=(Ax+B).e^{-3x}=(2x-1).e^{-3x}$

(b) $y''+y'-2y=0$. $\therefore (D^2+D-2)y=0$. $\therefore (D-1)(D+2)y=0$.

So $D=1$ or -2. $\therefore y=Ae^x+Be^{-2x}$. $\therefore y'=Ae^x-2Be^{-2x}$.

$y(0)=1 \Rightarrow A.e^0+Be^0=1 \Rightarrow A=1-B$

$y'(0)=7 \Rightarrow A-2B=7 \Rightarrow (1-B)-2B=7 \Rightarrow 3B=-6 \Rightarrow B=-2$

$\therefore A=1-(-2)=3$. Hence $y=Ae^x+Be^{-2x}=3e^x-2e^{-2x}$.

2. The homogeneous eq. is $y''+2y'-2y=0$. $\therefore (D^2+2D-2)y=0$

$\therefore D=[-2\pm\sqrt{4-(-8)}]/2=(-2\pm\sqrt{12})/2=(-2\pm2\sqrt{3})/2=-1\pm\sqrt{3}$

$\therefore \quad y_c=C_1.e^{(-1+\sqrt{3})x}+C_2.e^{(-1-\sqrt{3})x}$.

The non-homogeneous eq. is $y''+2y-2y=5\cos x \quad \ldots (**)$

So try $y_p=A\cos x+B\sin x$. Then

$y_p'=-A\sin x+B\cos x$ & $y_p''=-A\cos x-B\sin x$.

So the equation (**) becomes

$$(-A\cos x-B\sin x)+2(-A\sin x+B\cos x)-2(A\cos x+B\sin x)=5\cos x$$

$\therefore (-A+2B-2A)\cos x+(-B-2A-2B)\sin x=5\cos x$.

$\therefore \quad -3A+2B=5$ (1) and $-3B-2A=0$ (2)

From (2) $A=-3B/2$. Substituting in (1) gives

$$-3.(-3B/2)+2B=5 \Rightarrow (9/2+4/2)B=5 \Rightarrow B=10/13$$

$\therefore A=-3B/2=-15/13$ $\therefore y_p=(10\sin x-15\cos x)/13$.

$\therefore y=y_c+y_p=C_1.e^{(-1+\sqrt{3})x}+C_2.e^{(-1-\sqrt{3})x}+(10\sin x-15\cos x)/13$.

3. The homogeneous eq. is $y'' - y = 0$. So $(D^2-1)y = 0$

$\therefore (D-1)(D+1)y = 0$. $\therefore D = 1$ or -1. So $y_c = C_1 e^x + C_2 e^{-x}$.

The non-homog. eq is $y'' - y = 4x.e^x$. Since $4x$ is a polynomial of degree 1 and $\alpha = 1$ is a root of multiplicity 1, we should try $y_p = (Ax+B).x^1.e^x = (Ax^2+Bx).e^x$.

So $y_p' = (Ax^2+Bx).e^x + (2Ax+B).e^x = (Ax^2 + 2Ax + Bx + B)e^x$

$\therefore y_p'' = (Ax^2+2Ax+Bx+B).e^x + (2Ax+2A+B)e^x = (Ax^2+4Ax+2Bx+2A+2B)e^x$

$\therefore (Ax^2+4Ax+Bx+2A+2B)e^x - (Ax^2+Bx)e^x = 4x.e^x$.

$\therefore [(4A+B-B)x + (2A+2B)].e^x = 4x.e^x$

$\therefore 2A+2B = 0$ & $4A = 4 \Rightarrow A = 1$. So $B = -1$

$\therefore y_p = (x^2-x)e^x$. $\therefore y = y_c + y_p = C_1 e^x + C_2 e^{-x} + x(x-1).e^x$.

4(a) $(D^2-4D+5)^2 y = 3 + 5x.e^{2x}.\cos x$. $(D^2-4D+5)^2 y = 0$

$\therefore D = [-(-4) \pm \sqrt{16-20}]/2 = (4 \pm 2i)/2 = 2 \pm i$ (twice). So

$y_c = [(C_1 + C_2 x)\cos x + (C_3 + C_4 x).\sin x].e^{2x}$.

Since $5x$ is a polynomial of degree 1 and $2 \pm i$ are roots of the aux. eq. of multiplicity 2; and the root 0 corresponding to $3 = 3.e^{0x}$ is not present in y_c, the minimal form of y_p that we should try is

$y_p = A_0 + [(A_1 + A_2 x)\cos x + (B_1 + B_2 x).\sin x].x^2.e^{2x}$

(b) $D(D^2-1)(D^2+4)y = e^{-x} + 6\cos^2 x = e^{-x} + 6.\frac{1+\cos 2x}{2} = e^{-x} + 3 + 3\cos(2x)$.

$D(D^2-1)(D^2+4) = 0 \Rightarrow D(D-1)(D+1)(D-2i)(D+2i) = 0$. Thus

$D = 0, 1, -1, 2i$ or $-2i$. Hence

$y_c = C_1 + C_2 e^x + C_3 e^{-x} + C_4 \cos(2x) + C_5 \sin(2x)$.

Since $D = 0, -1, 2i$ & $-2i$ are roots of the aux. equation which corresponds to $3, e^{-x}$ & $3\cos(2x)$, the minimal y_p is

$y_p = A_0.x + A_1.x.e^{-x} + A_3.x.\cos(2x) + A_4.x.\sin(2x)$.

5(a) $\{f_1, f_2, f_3, f_4\}$ is linearly independent if $c_1 . f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) \equiv 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$.

(b) One $y_p = v_1 y_1 + v_2 y_2$ where y_1 & y_2 are two linearly independent solutions of the homogeneous equation and

$$v_1' = \begin{vmatrix} 0 & y_2 \\ F(x) & y_2' \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \& \quad v_2' = \begin{vmatrix} y_1 & 0 \\ y_1' & F(x) \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Homog. Eq. is $(D-1)^2 y = 0$: So take $y_1 = e^x$ & $y_2 = xe^x$.

$$\therefore \quad v_1' = \begin{vmatrix} 0 & xe^x \\ -e^x/x^2 & (x+1)e^x \end{vmatrix} \Big/ \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (e^{2x}/x)/e^{2x} = 1/x.$$

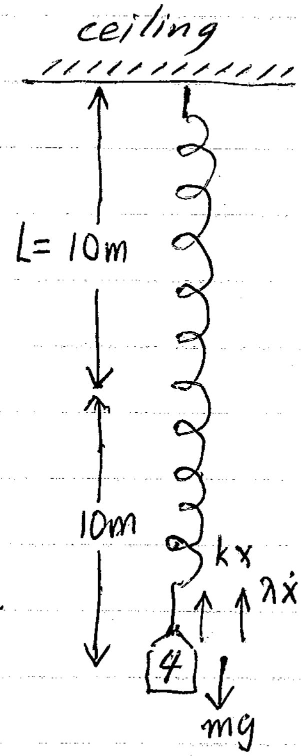
$$\therefore \quad v_1 = \int (1/x)\,dx = \ln(x).$$

$$v_2' = \begin{vmatrix} e^x & 0 \\ e^x & -e^x/x^2 \end{vmatrix} \Big/ \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (-e^{2x}/x^2)/e^{2x} = -1/x^2.$$

$$\therefore \quad v_2 = \int (-1/x^2)\,dx = 1/x.$$

$$\therefore \quad y_p = v_1 y_1 + v_2 y_2 = (\ln x).e^x + \frac{1}{x}.xe^x = (1+\ln x).e^x.$$

6. Let $x(t)$ = amount spring is extended at time t.



Then $m\ddot{x} = mg - \lambda\dot{x} - kx$, so $\ddot{x} + \frac{\lambda}{m}\dot{x} + \frac{k}{m}x = g$.

$\therefore \quad \ddot{x} + \dot{x} + \frac{5}{4}x = 10 \Rightarrow (D^2 + D + \frac{5}{4})x = 10.$

$D^2 + D + 5/4 = 0 \Rightarrow D = (-1 \pm \sqrt{1-5})/2 = -\frac{1}{2} \pm i.$

$\therefore \quad x_c(t) = e^{-t/2}(A\cos t + B\sin t).$

Try $x_p(t) = C$. Then $\dot{x}_p(t) = 0$ & $\ddot{x}_p(t) = 0$.

So $\ddot{x} + 2\dot{x} + 5x/4 = 10$ becomes $5C/4 = 10 \Rightarrow C = 8$.

$\therefore \quad x(t) = x_c(t) + x_p(t) = 8 + e^{-t/2}(A\cos t + B\sin t).$

$\dot{x}(t) = 0 - \frac{1}{2}e^{-t/2}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t).$

$= e^{-t/2}\left[(B - \frac{A}{2})\cos t - (A + \frac{B}{2})\sin t\right].$

Now $x(0) = 10$ and $\dot{x}(0) = 0$ because the spring was stretched by 10m & the body was released from rest at time $t=0$.

So $\quad 10 = 8 + e^{-0}(A\cos(0) + B\sin(0)) \Rightarrow 10 = 8 + A \Rightarrow A = 2.$

and $\quad 0 = e^{-0}\left[(B - \frac{A}{2})\cos(0) - (A + \frac{B}{2})\sin(0)\right] \Rightarrow B = 1.$

$\therefore \quad x(t) = 8 + e^{-t/2}(2\cos t + \sin t) = 8 + \sqrt{5}e^{-t/2}\left(\frac{2}{\sqrt{5}}\cos t + \frac{1}{\sqrt{5}}\sin t\right)$

$= 8 + \sqrt{5}e^{-t/2}.\cos(t - \alpha)$ where $\alpha = \tan^{-1}(1/2)$.