<u>*MAP 2302 - Differential Equations*</u> <u>*Florida International University*</u>

<u>*TEST 3 - Fall 2013*</u> <u>*TIME: 75 min.*</u>

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(15) 1. Starting with $\mathcal{L}\{e^{at}\}(s) = 1/(s-a)$, use the properties of the Laplace transform to find (a) $\mathcal{L}\{sin(t)\}(s)$ (b) $\mathcal{L}\{t^2 . sin(t)\}(s)$

(15) 2. Find the general solution of the linear ODE $x^2.y'' + x.y' - y = 4.x$ by first transforming it into a linear constant coefficient ODE in y and t.

(20) 3. For each of the following ODEs, find the indicial equation and the form of two linearly independent Frobenius series solution about $x_0 = 0$.

(a) $x^2.y'' + 2x.y' + (5/4 - x).y = 0.$

(b) $4x.y'' + 4.y' + \{(x-1)/x\}.y = 0.$

(15) 4. Find the first 5 non-zero terms of the power series solution of the ODE

$$y'' + x.y' - 2.y = 0 \quad \text{with} \quad y(0) = 2 \;\&\; y'(0) = 3.$$

(17) 5. Solve each of the following *IVPs*, by using the Laplace transform.

(a) $y'(t) + (1/2).y(t) = 3.e^{-t/2}$ with $y(0) = -2$.

(b) $y''(t) - 2.y'(t) + 2.y(t) = 0$ with $y(0) = 1$ & $y'(0) = 3$.

(18) 6. (a) Solve the following system of linear ODEs, by using the Laplace transform.

$$x'(t) - y(t) = 4 \quad \text{with} \quad x(0) = 2 \;\&\; y(0) = 0.$$
$$y'(t) - x(t) = 0$$

(b) Define what it means for 0 to be a *singular point* and what it means for 0 to be a *regular singular point* of the ODE $y'' + P_1(x).y' + P_2(x).y = 0.$

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1(a) $\mathcal{L}\{\sin t\}(s) = \mathcal{L}\{(e^{it} - e^{-it})/2i\}(s) = (1/2i).\left[\mathcal{L}\{e^{it}\}(s) - \mathcal{L}\{e^{-it}\}(s)\right]$

$$= \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{2i}\,\frac{(s+i)-(s-i)}{(s+i)(s-i)} = \frac{1}{s^2+1}$$

(b) $\mathcal{L}\{t^2\sin t\}(s) = (-1)^2 \frac{d^2}{ds^2}\left[\mathcal{L}\{\sin t\}(s)\right] = \frac{d}{ds}\left(\frac{d}{ds}\left[\frac{1}{s^2+1}\right]\right) = \frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right]$

$$= \frac{d}{ds}\left[(-2s)(s^2+1)^{-2}\right] = (-2).1.(s^2+1)^{-2} + (-2s).(-2).(s^2+1)^{-3}.\,2s$$

$$= \frac{-2(s^2+1)}{(s^2+1)^3} + \frac{8s^2}{(s^2+1)^3} = \frac{6s^2-2}{(s^2+1)^3} = \frac{2(3s^2-1)}{(s^2+1)^3}.$$

2. Let $x = e^t$ and $\Delta = \frac{d}{dt}$. Then $x^2y'' = \Delta(\Delta-1)y$ & $xy' = \Delta y$.
So $x^2y'' + xy' - y = 4x$ becomes $[\Delta(\Delta-1) + \Delta - 1]y = 4e^t$.
So $(\Delta^2 - 1)y = 4e^t$. Homog. Eq. is $(\Delta-1)(\Delta+1)y = 0$
$\therefore\ y_c = C_1e^t + C_2e^{-t} = C_1.x + C_2.x^{-1}$.
Try $y_p = Ate^t$. Then $\dot{y}_p = A(t+1)e^t$ & $\ddot{y}_p = A(t+2)e^t$.
So our ODE becomes $A(t+2)e^t - Ate^t = 4e^t$.
$\therefore\ 2Ae^t = 4e^t$. Hence $A = 2$. So $y_p = Ate^t = 2te^t =$
$2.(\ln x).x$. Hence $y = C_1x + C_2.x^{-1} + 2x.\ln(x)$.

3(a) The ODE is $x^2y'' + 2xy' + (5/4 - x)y = 0$. So the associated Cauchy-Euler ODE is $x^2y'' + 2xy' + (5/4)y = 0$. Hence the auxiliary equation is $\Delta(\Delta-1) + 2\Delta + 5/4 = 0$ where $\Delta = \frac{d}{dt}$ and $x = e^t$. So the indicial equation is: $r(r-1) + 2r + \frac{5}{4} = 0$
$\therefore\ r^2 + r + 5/4 = 0$. So $r = (-1 \pm \sqrt{1-5})/2 = (-1/2) \pm i$.
Since the roots are a pair of complex numbers, we get two linearly independent solutions of the form:

$y_1(x) = x^{-1/2}.\cos(\ln x).\sum_{n=0}^{\infty} a_n.x^n$ with $a_0 = 1$ and

$y_2(x) = x^{-1/2}.\sin(\ln x).\sum_{n=0}^{\infty} b_n.x^n$ with $b_0 = 1$.

$[r_1 - r_2 = (-\frac{1}{2}+i) - (-\frac{1}{2}-i) = 2i \notin \mathbb{N}]$

3(b) We have $4xy'' + 4y' + (x-1)y/x = 0$. So $4x^2y'' + 4xy' + (x-1)y = 0$

The associated Cauchy-Euler ODE is $4x^2y'' + 4xy' - y = 0$

$\therefore$ auxiliary equation is $4\Delta(\Delta-1) + 4\Delta - 1 = 0$. So $4\Delta^2 - 1 = 0$.

$\therefore$ indicial eq. is $4r^2 - 1 = 0$. $\therefore (2r-1)(2r+1) = 0$. So

$r_1 = 1/2$ and $r_2 = -1/2$. Since $r_1 - r_2 = 1$ which is a pos. integer, we get two linearly independent solutions of the form:

$$y_1(x) = x^{1/2} . \sum_{n=0}^{\infty} a_n . x^n \text{ with } a_0 = 1, \text{ and}$$

$$y_2(x) = A . y_1(x) . \ln(x) + x^{-1/2} . \sum_{n=0}^{\infty} b_n . x^n \text{ with } b_0 = 1.$$

Here A is a constant which may or may not be zero.

4. Let $y = \sum_{n=0}^{\infty} a_n . x^n$. Then $y' = \sum_{n=1}^{\infty} n . a_n . x^{n-1}$ and

$$y'' = \sum_{n=2}^{\infty} n(n-1) . a_n . x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} . x^n. \text{ So}$$

$y'' + xy' - 2y = 0$ becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} . x^n + x . \sum_{n=1}^{\infty} n . a_n . x^{n-1} - 2 . \sum_{n=0}^{\infty} a_n . x^n = 0$$

$$\therefore (2a_2 - 2a_0) . x^0 + \sum_{n=1}^{\infty} \{(n+2)(n+1) a_{n+2} + (n-2) . a_n\} . x^n = 0$$

$\therefore 2a_2 - 2a_0 = 0$ & $(n+2)(n+1) a_{n+2} + (n-2) a_n = 0$ for $n \geq 1$

But $y(0) = \sum_{n=0}^{\infty} a_n . 0^n = a_0$, so $a_0 = y(0) = 2$. (Note: $0^0 = 1$)

Also $y'(0) = \sum_{n=1}^{\infty} n . a_n . 0^{n-1} = a_1$, so $a_1 = y'(0) = 3$. Hence

$$a_2 = (2a_0)/2 = 2 \text{ and } a_{n+2} = -(n-2) a_n / (n+2)(n+1)$$

$$\therefore a_3 = a_{1+2} = -(1-2) a_1 / (1+2)(1+1) = 3/6 = 1/2$$

$$a_4 = a_{2+2} = -(2-2) a_2 / (2+2)(2+1) = 0$$

$$a_5 = a_{3+2} = -(3-2) a_3 / (3+2)(3+1) = -a_3/20 = -1/40.$$

So the first 5 non-zero terms of $y(x)$ are given by

$$y(x) = 2 + 3x + 2x^2 + x^3/2 + 0 . x^4 - x^5/40 + \cdots$$

5(a) $y'(t) + (1/2) y(t) = 3e^{-t/2}$. So $\mathcal{L}\{y'\} + (1/2)\mathcal{L}\{y\} = 3\mathcal{L}\{e^{-t/2}\}$

$\therefore s\mathcal{L}\{y\} - y(0) + 1/2 \mathcal{L}\{y\} = 3/(s+1/2)$. $\therefore (s+1/2)\mathcal{L}\{y\} = \frac{3}{s+\frac{1}{2}} + y(0)$

$\therefore \mathcal{L}\{y\} = \frac{3}{(s+1/2)^2} - \frac{2}{(s+1/2)}$. $\therefore y(t) = 3te^{-t/2} - 2e^{-t/2}$.

5(b) $y''(t) - 2y'(t) + 2.y(t) = 0$ with $y(0) = 1$ and $y'(0) = 3$

So $\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0$

$\therefore\ s^2\mathcal{L}\{y\} - s.y(0) - y'(0) - 2[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\} = 0$

$\therefore\ (s^2 - 2s + 2)\mathcal{L}\{y\} = s\,y(0) + y'(0) - 2y(0) = s + 3 - 2 = s + 1$

$\therefore\ \mathcal{L}\{y\} = \dfrac{s+1}{s^2-2s+2} = \dfrac{s-1}{(s-1)^2+1} + \dfrac{2}{(s-1)^2+1}$. So $y(t) = e^t\cos t + 2e^t\sin t$

6(a) $\left.\begin{array}{l} x'(t) - y(t) = 4 \\ y'(t) - x(t) = 0 \end{array}\right\}$ & $x(0) = 2$, $y(0) = 0$ $\quad \therefore\ \left.\begin{array}{l} \mathcal{L}\{x'\} - \mathcal{L}\{y\} = \mathcal{L}(4) \\ \mathcal{L}\{y'\} - \mathcal{L}\{x\} = 0 \end{array}\right\}$

$\therefore\ \left.\begin{array}{l} s\mathcal{L}\{x\} - x(0) - \mathcal{L}\{y\} = 4/s \\ s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{x\} = 0 \end{array}\right\}\begin{array}{l} (1) \\ (2) \end{array}$ From (2) $\mathcal{L}\{x\} = s\mathcal{L}\{y\} - y(0)$ $= s\mathcal{L}\{y\}$.

Substituting in (1) gives us $s.[s\mathcal{L}\{y\}] - 2 - \mathcal{L}\{y\} = 4/s$

$\therefore\ (s^2 - 1)\mathcal{L}\{y\} = 2 + \dfrac{4}{s} = \dfrac{2s+4}{s}$

$\therefore\ \mathcal{L}\{y\} = \dfrac{2s+4}{(s^2-1)s} = \dfrac{2s+4}{s(s-1)(s+1)} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{C}{s+1}$

Hence $2s + 4 = A(s-1)(s+1) + Bs(s+1) + Cs(s-1)$

Putting $s = 0$ gives us $2(0) + 4 = A(-1)(1) \Rightarrow A = -4$

Putting $s = 1$ gives us $2(1) + 4 = B(1)(1+1) \Rightarrow B = 3$

Putting $s = -1$ gives us $2(-1) + 4 = C(-1)(-1-1) \Rightarrow C = 1$

$\therefore\ \mathcal{L}\{y\} = \dfrac{-4}{s} + \dfrac{3}{s-1} + \dfrac{1}{s+1}$. So $y(t) = -4 + 3e^t + e^{-t}$.

Now we are given that $y'(t) - x(t) = 0$. So

$x(t) = y'(t) = (-4 + 3e^t + e^{-t})' = 3e^t - e^{-t}$.

[Check your answers by verifying that $x(0) = 2$ & $y(0) = 0$.]

(b) 0 is a singular point of $y'' + P_1(x)y' + P_2(x)y = 0$ if at least one of the two functions $P_1(x)$ & $P_2(x)$ is not analytic at $x=0$. 0 is a regular singular point if 0 is a singular point & both $x.P_1(x)$ & $x^2.P_2(x)$ are analytic at 0.