Ch. 1 - Basic Concepts & First Order ODEs

§1. Classification of ODEs - order and degree

An ordinary differential equation (ODE) is an equation involving at least one derivative of one or more dependent variables with respect to (w.r.t) to a single independent variable. A partial differential equation (PDE) is an equation involving at least one partial derivative of one or more dependent variables w.r.t. more than one independent variable.

Recall from Calculus that if $y = f(x)$ is a function of x, then we define $\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

We often write y' for dy/dx, y'' for $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, y''' for d^3y/dx^3, ... and $y^{(n)}$ for d^ny/dx^n.

Ex. 1 Let $y = e^{x^2}$. Then $y' = 2x.e^{x^2}$ and
$y'' = (2x)'.e^{x^2} + (2x).(e^{x^2})' = (2+4x^2)e^{x^2}$.

If $z = f(x,y)$ is a function of the independent variables x & y, then we define the partial derivatives of z w.r.t. x & y respectively, by

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and}$$

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}.$$ We also define

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right),$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \text{ and } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

Ex 2 Let $z = x^3 + \sin(y^2x)$. Then

$$\frac{\partial z}{\partial x} = 3x^2 + \cos(y^2x) \cdot \frac{\partial}{\partial x}(y^2x) = 3x^2 + y^2\cos(y^2x)$$

$$\frac{\partial z}{\partial y} = 0 + \cos(y^2x) \cdot \frac{\partial}{\partial y}(y^2x) = 2yx \cdot \cos(y^2x)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2y.1.\cos(y^2x) - 2yx.y^2 \sin(y^2x)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 0 + 2y.\cos(y^2x) - 2yx.y^2 \sin(y^2x)$$

Notice that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ for this example. This is not always true for an arbitrary function $z(x,y)$ but if $z(x,y)$ is "nice" (and we will always be dealing with nice functions), this will be true.

Ex 3 (a) $\frac{d^2y}{dx^2} + 3x.\left(\frac{dy}{dx}\right)^2 - 6x^2y^3 = e^x$ is an example of an ODE.

(b) $x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial z}{\partial x}.\left(\frac{\partial z}{\partial y}\right)^2 - z^2.\frac{\partial z}{\partial x} = x\cos(x^2y)$

is an example of a PDE.

We will not study PDEs in this course - but we will sometimes need to use partial derivatives.

<u>Classification of ODEs</u>

Any ODE can be written in the form

(*) $G(x, y, y', y'', \ldots, y^{(n)}) = 0$ for all $x \in I$

where G is a real function of the $n+2$ quantities $x, y, y', y'', \ldots, y^{(n)}$ and I is an interval known as the <u>interval of validity</u> of the ODE. Naturally we would like the interval of validity to be as big as possible. The ODE (*) is called an <u>implicit</u> ODE and its <u>order</u> is highest derivative that occurs non-trivially in it.

Def. An *explicit ODE of order n* is one that can be written in the form
$$y^{(n)} = F(x, y, y', y'', \ldots, y^{(n-1)}) \quad \text{for all } x \text{ in } I$$

An ODE is called *autonomous* if the independent variable x does not explicitly appear in it.

An ODE is called *polynomial-type* if it is of the form
$$P(y, y', \ldots, y^{(n)}) = 0 \quad \text{for all } x \text{ in } I$$
where P is a polynomial in $y, y', y'', \ldots, y^{(n)}$ with arb. func. of x as coeff. The *order* of this equation is, of course, the order of the polynomial P and the *degree* is the degree of the polynomial P.

Ex.4 (a) $(y'')^3 - xe^x \cdot \sin(y') + y^2 = \cos(x)$ is an *implicit* ODE of *order 2*. It does *not* have a degree because $\sin(y')$ is not a polynomial in y'

(b) $y''' - 3e^x \cdot y'' + y^2 \cdot (y') - x \cdot e^y = 0$ is an *explicit* ODE of order 3. It does *not* have a degree because e^y is not a polynomial in y.

(c) $y'' + 3y \cdot (y')^4 - y^3 = 0$ is an autonomous ODE because x does not explicitly appear in it. It is also an ODE of *order 2* and *degree 5* because the term y'' has degree 1, the term $3y \cdot (y')^4$ has degree 5, and the term $-y^3$ has degree 3. (The *degree* of a polynomial in several arguments is the maximum degree of all its terms.)

(d) $x^2 y'' - 3x^3 \cdot y^2 \cdot y' + e^x \cdot y^2 = 0$ is a polynomial-type ODE of order 2 and degree 3. The x^2, $3x^3$, & e^x do *not* count in the degree or order

Def. A linear ODE of order n is any ODE of the form $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)\cdot y = b(x)$

In other words a linear ODE of order n is a polynomial-type ODE of order n and degree 1.

Ex.5 (a) $y'' + x^2 \cdot y' + e^x \cdot y = \sin(x)$ is a linear equation of order 2.

(b) $x^3 y''' - 3xy' + 5y = x^3 e^x$ is an ODE which can be made into a linear ODE of order 3 just by dividing both sides by x^3.

(c) $y''' - 5y'' + 6y = 2e^x$ is a linear ODE of order 3 with constant coefficients

Before we go any further, let us try to solve a specific first-order ODE.

Ex.6 Find the general solution of the ODE

$$2y\frac{dy}{dx} + 8x = 0$$

Sol. Let us rewrite the equation as $2y\frac{dy}{dx} = -8x$

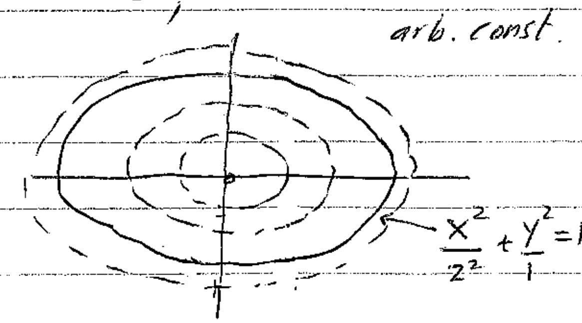
$$\therefore \int\left(2y\frac{dy}{dx}\right)dx = \int(-8x)\,dx$$

So $y^2 + C_1 = -4x^2 + C_2$ where C_1 & C_2 are arb. const.

$\therefore\ 4x^2 + y^2 = C_2 - C_1 = C$, where C is an arb. const.

$$\therefore\ \frac{x^2}{(2\sqrt{C})^2} + \frac{y^2}{(\sqrt{C})^2} = 1.$$

So for each positive C we get a solution.



§2. Classification of solutions & existence Theorem.

Recall that any ODE can be written in the form

(*) $G(x, y, y', y'', \ldots, y^{(n)}) = 0$ for all x in I,

where G is a real function of the $(n+2)$ arguments $x, y, y', \ldots, y^{(n)}$ and I is an interval.

Def. An explicit solution of the ODE (*) is any real function $f(x)$ such that $f^{(n)}(x)$ exists and

$$G(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0 \text{ for all } x \text{ in } I.$$

We usually write an explicit solution in the form $y = f(x)$.

An implicit solution of (*) is any relation $g(x,y)=0$ which defines at least one real-valued function $f(x)$ which is an implicit solution of (*).

Ex1. Consider the ODE $y'' - 4y = 0$ for all x in $(-\infty, \infty)$. The function $f(x) = e^{2x}$ is an explicit solution of $y'' - 4y = 0$. Why? Well, let's check.

$f(x) = e^{2x}$, $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$.

So $f''(x) - 4f(x) = 4e^{2x} - 4.e^{2x} = 0$ for all x in $(-\infty, \infty)$. So $y = f(x) = e^{2x}$ is indeed a solution of $y'' - 4y$.

There are other solutions to the ODE $y'' - 4y = 0$. For example, one can check that $y = 3e^{-2x}$ is another solution.

Ex.2 Consider the ODE $y.y' + 2x^3 = 0$ for all x in $(-1,1)$. Let $g(x,y) = x^4 + y^2 - 1$. Then $g(x,y) = 0$ is an implicit solution of $y.y' + 2x^3 = 0$.

Let us check. We have $x^4 + y^2 - 1 = 0$. So

$$y^2 = 1 - x^4. \quad \therefore \quad y = \pm\sqrt{1-x^4} \text{ on } (-1,1).$$

Let $f_1(x) = \sqrt{1-x^4}$ and $f_2(x) = -\sqrt{1-x^4}$. Then

$$f_1'(x) = [(1-x^4)^{1/2}]' = (1/2)(-4x^3).(1-x^4)^{-1/2}$$
$$= -2x^3.(1-x^4)^{-1/2}$$

So $f_1(x).f_1'(x) + 2x^3 = (1-x^4)^{1/2}(-2x^3)(1-x^4)^{-1/2}$

$$= -2x^3 + 2x^3 = 0$$

for all x in $(-1,1)$. So $f_1(x)$ is an explicit solution of $y.y' + 2x^3 = 0$. We can also check that $f_2(x)$ is also a solution of $y.y' + 2x^3 = 0$. Since $f_1(x)$ and $f_2(x)$ came from the relation $x^4 + y^2 - 1 = 0$, it follows that $x^4 + y^4 - 1 = 0$ is an implicit solution of $y.y' + 2x^3 = 0$.

Initial-value problems & Boundary value problems.

In general, an ODE will have an infinite number of solutions. If we want to pick out a particular solution, we must specify initial conditions or specify boundary values.

Ex.3 Consider the ODE $y'' - y' = 0$ with the initial conditions $y(0) = 1$ and $y''(0) = 3$. Since this is a second order ODE we only need to specify $y(0)$ & $y'(0)$ — because $y''(x)$ can be obtained directly from the ODE. Also $y'''(x)$ can be obtain from the ODE by differentiating w.r.t. x and so can $y^{(n)}(x)$ for all $n \geq 2$. What we want to find is $y(x)$. And then we can easily get $y'(x)$ also.

Sol. Let $v = y'$. Then $v' = y''$. So the equation $y'' - y' = 0$ becomes $v' - v = 0$. $\therefore \frac{dv}{dx} = v$.

$\therefore \frac{dv}{v} = dx$, $\therefore \int \frac{dv}{v} = \int dx$

$\therefore \ln(v) = x + C$,

$\therefore v = e^{x+C} = e^{C} \cdot e^{x} = A \cdot e^{x}$, where $A = e^{C}$

$\therefore \frac{dy}{dx} = y' = v = Ae^{x}$.

$\therefore dy = Ae^{x} \cdot dx$ $\therefore \int dy = \int Ae^{x} dx$

$\therefore y = Ae^{x} + B$ is the most general solution.

But $y(0) = 1$ and $y'(0) = 3$. so

$1 = A \cdot e^{0} + B$ and $3 = A \cdot e^{0}$

Hence $A = 3$ and since $1 = A + B$, $1 = 3 + B \Rightarrow B = -2$

$\therefore y = 3e^{x} - 2$.

Ex. 4 Find the solution of $y'' - y' = 0$ with the boundary conditions $y(0) = 3$ and $y(1) = e + 2$.

Sol. We have already seen that $y = Ae^{x} + B$ is the general solution of $y'' - y' = 0$. Since $y(0) = 3$ and $y(1) = e + 2$, we get

$3 = A \cdot e^{0} + B \Rightarrow 3 = A + B \Rightarrow B = 3 - A$

and $e + 2 = Ae^{1} + B \Rightarrow e + 2 = Ae + 3 - A$

$\therefore e - 1 = A(e - 1) \Rightarrow A = 1$. $\therefore B = 3 - A = 2$.

Hence $y = Ae^{x} + B = e^{x} + 2$.

In general initial-value problems are much more common and much easier to solve than boundary-value problems. For this and other reasons, we shall deal mostly with initial value problems.

Not every algebraic equation has a real solution. For example the equation $x^2-3x+5=0$ has no real solution because the discriminant $b^2-4ac = (-3)^2-4(1)(5) = 9-20 = -11 < 0$. The same is true of ODEs. The ODE $(y')^2+2=0$ has no real solution. We do not have any nice criterion (like the discriminant being ≥ 0), but for First Order ODEs, we can guarantee that there will be a solution of explicit ODEs under certain conditions.

(Thm. 1) Picard's existence theorem. Let $f(x,y)$ be a continuous function whose partial derivatives are also continuous on the open rectangle $(a,b)\times(c,d)$. Let $x_0 \in (a,b)$ and $y_0 \in (c,d)$. Then the ODE $dy/dx = f(x,y)$ has a unique solution $y=\varphi(x)$ with $y_0 = \varphi(x_0)$ in some interval $(x_0-h, x_0+h) \subseteq (a,b)$ for some sufficiently small $h>0$.

Ex.5 Show that the ODE $dy/dx = yx^2 + e^{xy}$ has a unique solution with $y(1)=3$ on some interval containing 1.

Sol. We have $f(x,y) = yx^2+e^{xy}$. So $\partial f/\partial x = 2xy + ye^{xy}$ and $\partial f/\partial y = x^2 + xe^{xy}$. All of these are continuous on $(-\infty,\infty)\times(-\infty,\infty)$. Since $1\in(-\infty,\infty)$ and $3\in(-\infty,\infty)$, it follows from Picard's Theorem that there exists a unique solution $y=\varphi(x)$ with $\varphi(1)=3$ in some interval $(1-h,1+h)\subseteq(-\infty,\infty)$ for a small enough $h>0$. (How we find this solution is an altogether different story.)

§3. Exact first-order ODEs & total differentials.

Consider the ODE $\frac{dy}{dx} = \frac{x-y}{x^2+y^2}$. This ODE is of the form $dy/dx = f(x,y)$, so we know from the Picard's existence theorem that it will have solutions once the required conditions are satisfied. Since $f(x,y) = (x-y)/(x^2+y^2)$ is undefined at $(0,0)$ we might, of course, have trouble there.

We can rewrite this equation as

$$(x^2+y^2)\,dy = (x-y)dx$$

and further more rearrange it as

$$(y-x).\,dx + (x^2+y^2)dy = 0$$

This is called the standard differential form of the ODE $dy/dx = (x-y)/(x^2+y^2)$.

Def. The standard differential form of a First-Order ODE is one of the form $M(x,y)\,dx + N(x,y)\,dy = 0$

Our goal for the rest of this chapter is to show how to solve as many ODEs of this form, as we can. In order to do this we need the concept of a total derivative of a function $F(x,y)$

Def. Let $F(x,y)$ be a function of x & y which is continuous and has continuous partial derivatives $\partial F/\partial x$ & $\partial F/\partial y$ on some open rectangle $(a,b)\times(c,d)$. We define the total derivative of F by $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$.

Ex.1 Let $F(x,y) = xy^2 + \sin(x^2y)$. Find dF.

Sol. $F(x,y) = xy^2 + \sin(x^2y)$. Then

$\partial F/\partial x = 1.y^2 + \cos(x^2y).\,\partial(x^2y)/\partial x = y^2 + 2xy\cos(x^2y)$,

& $\partial F/\partial y = x.2y + \cos(x^2y).\,\partial(x^2y)/\partial y = 2xy + x^2\cos(x^2y)$.

$$\therefore\ dF = \frac{\partial F}{\partial x}.dx + \frac{\partial F}{\partial y}.dy$$

$$= \left[y^2 + 2xy\cos(x^2y)\right]dx + \left[2xy + x^2\cos(x^2y)\right]dy.$$

Ex.2 Find the general solution of the First Order ODE

$$(3x^2+2xy)\,dx + (x^2-6y)\,dy = 0 \quad \cdots \quad (*)$$

Sol. We will look for a function $F(x,y)$ such that

$$dF = (3x^2+2xy)\,dx + (x^2-6y)\,dy.$$

From $(*)$ we will get $dF = 0$, so $F(x,y) = C_1$ where C_1 is an arbitrary constant. Now

$$dF = \left(\frac{\partial F}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y}\right)dy.$$

So $\dfrac{\partial F}{\partial x} = 3x^2+2xy$ and $\dfrac{\partial F}{\partial y} = x^2-6y$.

$$\therefore\ F(x,y) = \int(3x^2+2xy)\,\partial x = x^3 + x^2y + \varphi(y)$$

where $\varphi(y)$ is an arbitrary function of y. [This is because we are integrating w.r.t. ∂x. A quick check shows that $\frac{\partial}{\partial x}(x^3+x^2y+\varphi(y)) = 3x^2+2xy$.]

$$\therefore\ \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[x^3+x^2y+\varphi(y)\right] = 0 + x^2 + \varphi'(y).$$

But $\dfrac{\partial F}{\partial y} = x^2-6y$ from above. So

$$x^2 + \varphi'(y) = x^2 - 6y, \quad \therefore\ \varphi'(y) = -6y$$

$$\therefore\ \varphi(y) = \int -6y.dy = -3y^2 + C_2.$$

Hence $F(x,y) = x^3+x^2y+\varphi(y) = x^3+x^2y-3y^2+C_2$

So the gen. sol. will be $x^3+x^2y-3y^2+C_2 = C_1$

We write this as $x^3+x^2.y-3y^2 = C$ where $C = C_1 - C_2$ is an arb. const.

Ex.3 Find the general solution of the ODE

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0 \quad \cdots (*)$$

Attempted Sol. Again, we will look for a function $F(x,y)$ with $dF = (3y+4xy^2)dx + (2x+3x^2y)dy$. Then from $(*)$, $dF = 0$ and we will get a solution $F(x,y) = C$. Now $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$. So

$$\frac{\partial F}{\partial x} = 3y + 4xy^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = 2x + 3x^2y$$

$$\therefore\ F = \int (3y+4xy^2)\,\partial x = 3xy + 2x^2y^2 + \varphi(y).$$

$$\therefore\ \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(3xy + 2x^2y^2 + \varphi(y)\right) = 3x + 4x^2y + \varphi'(y)$$

But $\frac{\partial F}{\partial y} = 2x + 3x^2y$. So

$$3x + 4x^2y + \varphi'(y) = 2x + 3x^2y.$$

But this means we must have

$$3x = 2x, \quad 4x^2y = 3x^2y, \quad \text{and} \quad \varphi'(y) = 0.$$

But the first two equations are not valid for all x, y.
So something must have gone wrong.

What went wrong was that we assume that there existed such a function $F(x,y)$ — when in reality such a function does not exist. This is because the ODE $(*)$ in Ex.3 was not exact, whereas the ODE in Ex.2 was exact

Def. The ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in an open rectangle $R = (a,b) \times (c,d)$ if we can find a function $F(x,y)$ such that

$$dF = M(x,y)\,dx + N(x,y)\,dy \quad \text{for each } \langle x,y \rangle \text{ in } R.$$

Qu. How can we easily tell if an a first-order ODE is exact?

Ans. Well, according to the definition we have to find a function $F(x,y)$ such that $dF = M(x,y)dx + N(x,y)dy$. But this is not an easy task. Fortunately, we have a criterion that works whenever the functions $M(x,y)$ & $N(x,y)$ are continuous.

Theorem 2 (Exactness criteria)
Let $M(x,y)$ and $N(x,y)$ be continuous functions on some open rectangle $R = (a,b) \times (c,d)$. Then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all $\langle x,y \rangle$ in R if and only if the ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$ is exact in R.

Ex. 4 Show that the ODE

$$[3x^2 \sin(y) + 2xy]\,dx + [x^3\cos(y) + x^2 - 10y]\,dy = 0$$

is exact in $R = (-\infty,\infty)\times(-\infty,\infty)$ & find its gen. sol.

Sol. Let $M(x,y) = 3x^2 \cdot \sin(y) + 2xy$ and

$N(x,y) = x^3 \cdot \cos(y) + x^2 - 4y^3$.

Then $\frac{\partial M}{\partial y} = 3x^2 \cdot \cos(y) + 2x$ and

$\frac{\partial N}{\partial x} = 3x^2 \cdot \cos(y) + 2x - 0$.

So $\partial M/\partial y = \partial N/\partial x$ for all $\langle x,y\rangle$ in R.

$\therefore$ the ODE is exact. So we can find a function $F(x,y)$ such that $dF = M(x,y)dx + N(x,y)dy$.

$\therefore \frac{\partial F}{\partial x} = 3x^2 \cdot \sin(y) + 2xy$ & $\frac{\partial F}{\partial y} = x^3\cos y + x^2 - 4y^3$.

So $F = \int (3x^2 \cdot \sin(y) + 2xy)\,\partial x = x^3 \sin(y) + x^2 y + \varphi(y)$

$\therefore \frac{\partial F}{\partial y} = x^3 \cdot \cos(y) + x^2 + \varphi'(y)$.

But $\frac{\partial F}{\partial y} = x^3 \cdot \cos y + x^2 - 4y^3$. $\therefore \varphi'(y) = -4y^3$

So $\varphi(y) = -y^4$. Hence $x^3 \cdot \cos(y) + x^2 - y^4 = C$ is the gen. sol.

<u>Qu:</u> What can we do if a First Order ODE is not exact.

<u>Ans:</u> Well, we can try to make it exact by multiplying throughout by a suitable function.

<u>Def.</u> Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be an ODE which is <u>not</u> exact. A function $\mu(x,y)$ is called an <u>integrating factor</u> of this ODE if the modified ODE

$$\mu(x,y)M(x,y)\,dx + \mu(x,y).N(x,y)\,dy = 0$$

is exact. (The term "integrating" is used because in the old days, they often spoke of integrating an ODE instead of solving it.)

<u>Ex. 4</u> Show that $\mu(x,y) = x^2y$ is an integrating factor of the ODE $(3y+4xy^2)\,dx + (2x+3x^2y)\,dy = 0$ $(*)$ and then find the general solution

<u>Sol.</u> Let $M(x,y) = 3y+4xy^2$ & $N(x,y) = 2x+3x^2y$. Then $\partial M/\partial y = 3+8xy$ and $\partial N/\partial x = 2+6xy$. So $\partial M/\partial y \neq \partial N/\partial x$

Hence $(*)$ is <u>not</u> exact. But

$$\partial(\mu M)/\partial y = \frac{\partial}{\partial y}(3x^2y^2+4x^3y^3) = 6x^2y+12x^3y^2 \text{ and}$$

$$\partial(\mu N)/\partial x = \frac{\partial}{\partial x}(2x^3y+3x^4y^2) = 6x^2y+12x^3y^2.$$

Hence $\mu.M(x,y)\,dx + \mu.N(x,y)\,dy$ is exact.

$\therefore$ $\mu(x,y) = x^2y$ is an integrating factor of $(*)$

Put $\frac{\partial F}{\partial x} = \mu M = 3x^2y^2+4x^3y^3$ & $\frac{\partial F}{\partial y} = \mu N = 2x^3y+3x^4y^2$

Then $F = \int(3x^2y^2+4x^3y^3)\,\partial x = x^3y^2+x^4y^3+\varphi(y)$

$\therefore$ $\partial F/\partial y = 2x^3y+3x^4y^2+\varphi'(y)$.

But $\partial F/\partial y = 2x^3y+3x^4y^2$. So $\varphi'(y) = 0$. $\therefore$ $\varphi(y) = C_2$

$\therefore$ $F(x,y) = x^3y^2+x^4y^3+\varphi(y) = x^3y^2+x^4y^3+C_2$

Since $dF = 0$ by $(*)$, $F(x,y) = C_1$. Hence

$x^3y^2+x^4y^3+C_1 = C_2$ So $x^3y^2+x^4y^3 = C$ is gen. sol.

We can write this as $x^3y^2(xy+1) = C$.

§4. Separable ODEs & ODEs with homogeneous coefficients.

It might appear that we can solve any first-order ODE — all we have to do is to just find an integrating factor. Unfortunately, there is no general method of finding integrating factors in all cases. Yes, the integrating factors always exist, but we can only find them in very special cases. We will study these cases in the rest of this chapter.

Def. The first-order ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is separable if we can find function $a(x)$, $b(y)$, $f(x)$ and $g(y)$ such that $M(x,y) = a(x).b(y)$ & $N(x,y) = f(x).g(y)$.

Fact: $\mu(x,y) = 1/[b(y).f(x)]$ is an integrating factor of the ODE $a(x).b(y).dx + f(x).g(y)\,dy = 0$. $(*)$

Proof Multiply $(*)$ throughout by $\mu(x,y)$. Then we will get

$$\frac{a(x)}{f(x)}.dx + \frac{g(y)}{b(y)}\,dy = 0 \quad \ldots (**)$$

Since $\frac{\partial}{\partial y}\left(\frac{a(x)}{f(x)}\right) = 0$ & $\frac{\partial}{\partial x}\left(\frac{g(y)}{b(y)}\right) = 0$, this new ODE $(**)$ is exact. $\therefore$ $\mu(x,y)$ is an integrating factor.

Ex. 1 Find the general solution of the ODE

$$(x+2).y.dx - x^2(4y^2+1).dy = 0$$

Sol. Divide by $y.x^2$ to get $\frac{x+2}{x}dx - \frac{4y^2+1}{y}dy = 0$

So $\left(\frac{1}{x} + \frac{2}{x^2}\right)dx = \left(4y + \frac{1}{y}\right)dy$

$\therefore \int\left(\frac{1}{x} + 2x^{-2}\right)dx = \int(4y + y^{-1})\,dy$

$\therefore \ln(x) - \frac{2}{x} = 4y + \ln(y) + C$ is the gen. solution.

Ex.2 Find the general solution of the ODE

$$2x.\cos y.\, dx - (x^2+1).\sin y.\, dy = 0 \quad \text{with } y(0) = \frac{\pi}{3}.$$

Sol We have $\quad 2x.\cos(y)\, dx = x^2+1.\sin(y)\, dy$

$$\therefore \quad \frac{2x}{x^2+1}\, dy = \frac{\sin y}{\cos y}\, dy = \tan(y)\, dy$$

$$\therefore \quad \int \frac{2x}{x^2+1}\, dx = \int \tan(y)\, dy$$

$$\therefore \quad \ln(x^2+1) = \ln(\sec y) + C.$$

$$\therefore \quad x^2+1 = e^{\ln(\sec y)+C} = e^C.\sec(y) = A.\sec(y)$$

But $y = \pi/3$, when $x = 0$. So

$$0^2+1 = A.\sec(\pi/3) = A.2.$$

$\therefore \quad 2A = 1$ and $A = \frac{1}{2}$. Hence $x^2+1 = \frac{1}{2}\sec(y)$.

So the solution is $\sec(y) = 2(x^2+1)$.

ODEs with homogeneous coefficients

Def. A function $f(x,y)$ is homogeneous of degree k if $f(\lambda x, \lambda y) = \lambda^k f(x,y)$.

Ex.3 Determine which of the following functions are homogeneous and, if they are, find their degree of homogeneity.

a) $f(x,y) = x^3 + 3x^2y + y^4/x$

b) $g(x,y) = e^x.e^{2y}$

c) $h(x,y) = x^2e^{x/y} + xye^{3y/x}$.

Sol. (a) $f(\lambda x, \lambda y) = (\lambda x)^3 + 3(\lambda x)^2(\lambda y) + (\lambda y)^4/(\lambda x)$

$$= \lambda^3.x^3 + \lambda^3.3x^2y + \lambda^3.y^4/x$$

$$= \lambda^3(x^3 + 3x^2y + y^4/x) = \lambda^3.f(x,y)$$

$\therefore \ f(x,y)$ is homogeneous of degree 3.

(b) $g(\lambda x, \lambda y) = e^{\lambda x} \cdot e^{2\lambda y} = (e^x)^\lambda \cdot (e^{2y})^\lambda = (e^x \cdot e^{2y})^\lambda$
$= g(x,y)^\lambda \neq \lambda^k \cdot g(x,y)$ for any k.
So $g(x,y)$ is not a homogeneous function

(c) $h(\lambda x, \lambda y) = (\lambda x)^2 \cdot e^{\lambda x/\lambda y} + (\lambda x)(\lambda y) \cdot e^{3\lambda y/\lambda x}$
$= \lambda^2 \cdot (x^2 e^{x/y} + xy \cdot e^{3y/x}) = \lambda^2 \cdot h(x,y)$
$\therefore$ $h(x,y)$ is a homog. function of degree 2.

Def. The first-order ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is an ODE with homogeneous coefficients of the same degree if $M(x,y)$ & $N(x,y)$ are homogeneous functions of the same degree. We usually just say that the ODE is homogeneous.

Theorem 3: If $M(x,y)dx + N(x,y)\,dy = 0$ is a homog. ODE, then it can be written in the form

$$dy/dx = f(y/x) \quad \cdots (*)$$

and the substitution $y = xv$ will transform $(*)$ into a separable ODE involving v and x.

Ex. 3 Find the general solution of the ODE

$$(3y^2 + 2x^2)\,dx - xy\,dy = 0$$

Sol. First of all this is a homogenous ODE because $(3y^2+2x^2)$ & (xy) are homog. functions of deg. 2.

Now $(xy)dy = (3y^2 + 2x^2)\,dx$

So $\dfrac{dy}{dx} = \dfrac{3y^2+2x^2}{xy} = 3\left(\dfrac{y}{x}\right) + 2\left(\dfrac{x}{y}\right)$

Now put $y = xv$. Then $\dfrac{dy}{dx} = \dfrac{d}{dx}(xv) = 1 \cdot v + x\dfrac{dv}{dx}$

So $v + x\dfrac{dv}{dx} = 3\left(\dfrac{y}{x}\right) + 2\left(\dfrac{x}{y}\right) = 3v + \dfrac{2}{v}$.

$\therefore$ $x\dfrac{dv}{dx} = 2v + \dfrac{2}{v}$ which is a separable ODE

$\therefore \quad x.dv/dx = 2v + 2/v = 2(v^2+1)/v$

$\therefore \quad \dfrac{2v\,dv}{v^2+1} = \dfrac{4dx}{x}$, so $\displaystyle\int \frac{2v\,dv}{v^2+1} = \int \frac{4dx}{x}$

$\therefore \quad \ln(v^2+1) = 4\ln(x) + C$

$\therefore \quad v^2+1 = e^{4\ln x + C} = e^C . e^{\ln(x^4)} = A . x^4$

$\therefore \quad v^2 = Ax^4 - 1$. So $\left(\frac{y}{x}\right)^2 = Ax^4 - 1$

$\therefore \quad y^2 = Ax^6 - x^2$ is the general solution.

Ex. 4 Find the solution of the ODE

$(y + 2\sqrt{xy})\,dx - x\,dy = 0$ with $y(1) = 9$.

Sol. Again we can see that $(y + 2\sqrt{xy})$ & x are both homog. functions of degree 1. So the ODE is homogeneous. Now $x\,dy = (y + 2\sqrt{xy})\,dx$

$\therefore \quad dy/dx = (y + 2\sqrt{xy})/x = (y/x) + 2\sqrt{y/x}$.

Put $y = xv$. Then $v = y/x$ & $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore \quad v + x.dv/dx = (y/x) + 2\sqrt{y/x} = v + 2\sqrt{v}$

$\therefore \quad x . dv/dx = 2\sqrt{v}$. $\therefore \quad \dfrac{dv}{2\sqrt{v}} = \dfrac{dx}{x}$

$\therefore \displaystyle\int \frac{1}{2} v^{-1/2}\,dv = \int x^{-1}\,dx$

$\therefore \quad v^{1/2} = \ln x + C$.

$\therefore \quad (y/x)^{1/2} = \ln(x) + C$. But $y(1) = 9$.

$\therefore \quad (9/1)^{1/2} = \ln(1) + C \Rightarrow C = 3$.

$\therefore \quad (y/x)^{1/2} = \ln(x) + 3$.

$\therefore \quad y/x = [3 + \ln(x)]^2$

$\therefore \quad y = x . [3 + \ln(x)]^2$ is the solution.

§5. Linear First-Order ODEs & Bernoulli ODEs.

Recall that a linear first-order ODE was one of the form $y' + a_0(x).y = b(x)$. It is more convenient to write this equation in the form $y' + p(x).y = q(x)$.

Theorem 4 (Integrating factor for Linear first-order ODEs)
The linear ODE $y' + p(x).y = q(x)$ has $e^{\int p(x)dx}$ as an integrating factor.

Proof: Let us write $dy/dx + p(x).y = q(x)$ in its differential form. We have $-dy/dx = p(x).y - q(x)$

So $[p(x).y - q(x)]dx = -dy$

$$\therefore \underbrace{[p(x)y - q(x)]}_{M}dx + \underbrace{1.}_{N}dy = 0$$

$\therefore \dfrac{\partial M}{\partial y} = p(x)$ & $\dfrac{\partial N}{\partial x} = 0$. Hence the linear ODE will not be exact unless $p(x) \equiv 0$. Let $\mu(x) = e^{\int p(x)dx}$

Then $\mu(x).M.dx + \mu(x).N.dy = 0$, becomes

$$\underbrace{e^{\int p(x)dx}.[p(x).y - q(x)]}_{M_1}dx + \underbrace{e^{\int p(x)dx}.1}_{N_1}.dy = 0$$

$\therefore \dfrac{\partial M_1}{\partial y} = e^{\int p(x)dx}.[p(x).1 - 0]$ & $\dfrac{\partial N_1}{\partial x} = p(x).e^{\int p(x)dx}$

Hence $\partial M_1/\partial y = \partial N_1/\partial x$ & so $\mu(x) = e^{\int p(x)dx}$ will be an integrating factor of $y' + p(x).y = q(x)$.

Ex.1 Find the general solution of the ODE

$$y' + 2.y = 6e^{x}$$

Sol. Here $p(x) = 2$. So integrating factor $= e^{\int 2dx} = e^{2x}$.

So $\quad e^{2x} \cdot \frac{dy}{dx} + e^{2x} \cdot 2y = e^{2x} \cdot 6e^{x}$.

$\therefore \quad \frac{d}{dx}\left(y \cdot e^{2x}\right) = 6e^{3x}$

this will always be y. (integrating factor)

$\therefore \quad y \cdot e^{2x} = \int 6 \cdot e^{3x}\, dx = 2e^{3x} + C$

$\therefore \quad y = e^{-2x}\left(2e^{3x} + c\right) = Ce^{-2x} + 2e^{x}$.

__Ex.2__ Find the solution of the ODE

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = 4x + 6 \qquad \text{with } y(1) = 2.$$

__Sol.__ This is a linear first-order ODE, so the integrating factor will be $e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln(x^2)} = x^2$

$\therefore \quad x^2 \cdot \frac{dy}{dx} + x^2 \cdot \frac{2}{x} y = x^2(4x + 6)$

$\therefore \quad x^2 \cdot \frac{dy}{dx} + 2x \cdot y = 4x^3 + 6x^2$

$\therefore \quad \frac{d}{dx}\left(y \cdot x^2\right) = 4x^3 + 6x^2$

the integrating factor

$\therefore \quad y \cdot x^2 = \int (4x^3 + 6x^2)\, dx = x^4 + 2x^3 + C$

But $y(1) = 2$. So $\quad 2 \cdot (1)^2 = (1)^4 + 2(1)^3 + C$

$\therefore \; C = -1$. So $\quad y \cdot x^2 = x^4 + 2x^3 - 1$

$\therefore \quad y = x^2 + 2x - x^{-2}$.

__Def.__ A __Bernoulli ODE__ is one of the form

$$y' + p(x) \cdot y = q(x) \cdot y^{\alpha}.$$

__Note__ Any Bernoulli ODE can be transformed into a linear ODE by putting $v = y^{1-\alpha}$. If $\alpha = 0$ or 1, then the Bernoulli ODE is already linear. And if $\alpha = 1$, the Bernoulli ODE is separable. So if $\alpha = 0$ or 1, we don't waste our time trying to transform it.

Theorem 5 Suppose $\alpha \neq 0$ or 1. Then the Bernoulli ODE $dy/dx + p(x).y = q(x).y^{\alpha}$ $(*)$
can be transformed into a linear ODE by putting $v = y^{1-\alpha}$.

Proof: Multiply both sides of the ODE $(*)$ by $(1-\alpha)y^{-\alpha}$.
Then $(1-\alpha)y^{-\alpha}\frac{dy}{dx} + (1-\alpha).y^{-\alpha}.y.p(x) = q(x).(1-\alpha)$
Now let $v = y^{1-\alpha}$. Then $\frac{dv}{dx} = \frac{dv}{dy}.\frac{dy}{dx} = (1-\alpha)y^{-\alpha}.\frac{dy}{dx}$

So we get $\frac{dv}{dx} + (1-\alpha)y^{1-\alpha}.p(x) = (1-\alpha).q(x)$
$\therefore \frac{dv}{dx} + (1-\alpha).p(x).v = (1-\alpha).q(x)$
which is a linear ODE in v and x.

Ex. 3 Find the general solution of the ODE
$$\frac{dy}{dx} + y = e^{x}.y^{3}$$

Sol. Multiply throughout by $(1-3)y^{-3} = (-2).y^{-3}$. Then
$$-2.y^{-3}.\frac{dy}{dx} + (-2)y^{-3}.y = e^{x}.(-2).y^{-3}.y^{3}$$
$$\therefore -2y^{-3}\frac{dy}{dx} - 2y^{-2} = -2e^{x}$$
Put $v = y^{1-3} = y^{-2}$. Then $\frac{dv}{dx} = \frac{dv}{dy}.\frac{dy}{dx} = -2.y^{-3}.\frac{dy}{dx}$

So we get $\frac{dv}{dx} - 2v = -2e^{x}$.
$\therefore$ integrating factor $= e^{\int -2dx} = e^{-2x}$.
$$\therefore e^{-2x}\frac{dv}{dx} - 2e^{-2x}.v = -2.e^{x}.e^{-2x}$$
$$\therefore \frac{d}{dx}\left(ve^{-2x}\right) = -2e^{-x}$$
$$\therefore ve^{-2x} = \int -2e^{-x}dx = 2e^{-x} + C$$
$$\therefore v = e^{2x}(C + 2e^{-x}) = Ce^{2x} + 2e^{x}$$
$$\therefore y^{-2} = Ce^{2x} + 2e^{x}. \quad \therefore y^{2}(Ce^{2x} + 2e^{x}) = 1.$$

Ex4: Find the general solution of the ODE

$$dy/dx + 2y = 4 . e^x . y^{1/2}$$

Sol Multiply both sides by $(1-1/2)y^{-1/2} = (1/2)y^{-1/2}$.

Then $\frac{1}{2}y^{-1/2} \frac{dy}{dx} + 2 \cdot \frac{1}{2} y^{-1/2} . y = \frac{4}{2} . e^x$

$\therefore \quad \frac{1}{2}y^{-1/2} \frac{dy}{dx} + y^{1/2} = 2e^x$

Put $v = y^{1-1/2} = y^{1/2}$. Then $\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$.

So our ODE becomes

$$\frac{dv}{dx} + v = 2 . e^x$$

The integrating factor $= e^{\int 1 dx} = e^x$.

$\therefore \quad e^x \frac{dv}{dx} + e^x v = 2e^x . e^x$

$\therefore \quad \frac{d}{dx}(v . e^x) = 2e^{2x}$

$\therefore \quad v . e^x = e^{2x} + C$

$\therefore \quad v = e^{-x}(e^{2x} + C) = e^x + Ce^{-x}$

$\therefore \quad y^{1/2} = e^x + Ce^{-x}$

$\therefore \quad y = (e^x + Ce^{-x})^2$.

Thm 6 (Finding integrating factors in special cases)

Consider the non-exact ODE $M(x,y)dx + N(x,y)dy = 0 \quad \ldots (*)$

(a) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then $e^{\int f(x)dx}$ will be an integrating factor of $(*)$

(b) If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$, then $e^{-\int g(y)dy}$ will be an integrating factor of $(*)$

Ex.5 Find an integrating factor of the ODE

$$\underbrace{(2x^2+y)}_{M} dx + \underbrace{(x^2y - x)}_{N} dy = 0$$

Sol. Here $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 - (2xy - 1) = 2(1 - xy)$.

So $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x(xy-1)} \cdot 2(1-xy) = \frac{-2}{x}$

So an integrating factor will be $e^{\int \frac{-2}{x}dx} = e^{-2\ln x}$
$= e^{\ln(x^{-2})} = x^{-2}$.

Ex.6 Find an integrating factor of the ODE

$$(xy^2+y)dx + (y^3-x)dy = 0$$

Sol. Here $\partial M/\partial y - \partial N/\partial x = (2xy+1) - (-1) = 2(xy+1)$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y^3-x} \cdot 2(xy+1) \neq$ any $f(x)$

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(xy+1)y} \cdot 2(xy+1) = \frac{2}{y}$.

$\therefore$ $e^{-\int \frac{2}{y}dy}$ will be an integrating factor.

$e^{-\int \frac{2}{y}dy} = e^{-2\ln y} = e^{\ln(y^{-2})} = y^{-2}$.

So an integrating factor will be y^{-2}.

END OF Ch. 1