Chapter 2 - Applications of First Order ODEs

Differential equations were introduced for the expressed purpose of solving problems that occur in the natural world. We begin with some problems that the great mathematician & physicist Isaac Newton tackled.

§1. Motion under gravity without & with air resistance

Newton's 2nd law of motion states that $\frac{d(mv)}{dt} = kF$. Here m = mass of the particle, v = velocity of the particle, F = resultant force on the particle, and k is a constant. If we choose the metric system as our units – length in metres (m), mass in kilograms (kg), and time in seconds (s), then $k=1$. (We will only use the metric system in these notes.) So $\frac{d(mv)}{dt} = F$ is Newton's 2nd law in metric units.

Ex. 1 (Vertical motion with no air resistance). A metal ball was thrown vertically upwards from the ground. At time $t = 0$ seconds, the ball was moving upwards with velocity $20\,ms^{-1}$ and was $25\,m$ above the ground.

(a) Find the maximum height above the ground that the ball reaches.

(b) How long will it take for the ball to hit the ground?

(c) What was the velocity with which the ball was initially thrown?

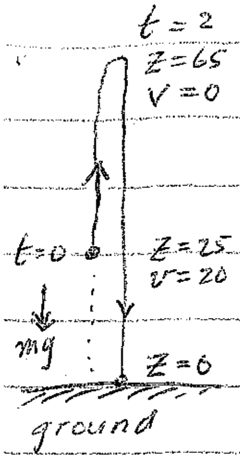
[Use $g = 10\,ms^{-2}$]

Sol. Let $z(t)$ = distance from the ground of the ball at time t.

Then $v(t) = \frac{dz}{dt}$ and $v(0) = 20\,ms^{-1}$ and $z(0) = 25\,m$.

Now $\frac{d(mv)}{dt} = -mg$ So $m\frac{dv}{dt} = -m(10)$

$\therefore\ dv = -10\,dt$. Hence $v = -10t + C_1$



But $v(0) = 20$. So $20 = -10(0) + C_1 \Rightarrow C_1 = 20$

$\therefore\ v(t) = -10t + 20$. Now $v(t) = \frac{dz}{dt}$

So $\frac{dz}{dt} = -10t + 20$ $\therefore\ dz = (-10t + 20)\,dt$

$\therefore\ z(t) = -5t^2 + 20t + C_2$

But $z(0) = 25$. So $25 = -5(0)^2 + 20(0) + C_2 \Rightarrow C_2 = 25$

$\therefore\ z(t) = -5t^2 + 20t + 25 = -5(t^2 - 4t - 5)$.

(a) The ball will reach its greatest height when $v(t) = 0$. So $-10t + 20 = 0 \Rightarrow t = 2$ sec.

$\therefore$ greatest height $= z(2) = -5(2^2 - 4(2) - 5) = 45\,m$

(b) The ball will hit the ground when $z(t) = 0$. So

$-5(t^2 - 4t - 5) = 0 \Rightarrow -5(t+1)(t-5) = 0 \Rightarrow t = 5$ or -1.

So the ball will hit the ground at time $t = 5$ sec.

The "-1" represents the time the was released from the ground. So ball was thrown upwards at time $t = -1$ sec.

(c) The ball was initial thrown with velocity $v(-1)$

$= -10(-1) + 20 = 30\,ms^{-1}$. [Since there is no air resistance, no energy will be lost — so the ball should hit the ground with the same speed but in the downward direction. Let's check: $v(5) = -10(5) + 20 = -30\,ms^{-1}$.]

Ex. 2 (Falling body problem with resistance proportional to $|v|$)

An object of mass 3kg falls from rest towards the earth from a great height. If the air resistance is $\lambda|v|$, where $\lambda = \frac{3}{10}\,kg\,s^{-1}$. After t seconds since it was released, what is

(a) the velocity of the object & the maximum value it can be.

(b) the distance the object has fallen. [Use $g = 10\,ms^{-2}$]

Sol. Let $t = 0$ be the time when the object was released and $z(t)$ = the distance the object falls in t seconds.

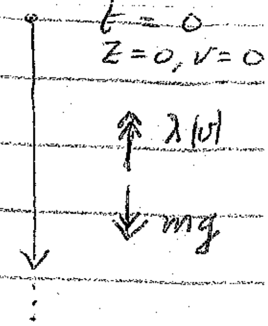
Then $z(0)=0$, $v(t)=\frac{dz}{dt}$, and $v(0)=0$. Also

(a) $\frac{d}{dt}(mv) = mg - \lambda v$. So

$$3\frac{dv}{dt} = 3(10) - \frac{3}{10}v.$$

$$\therefore \quad \frac{dv}{dt} = 10 - \frac{v}{10} = \frac{100-v}{10}$$

$$\therefore \quad \frac{-dv}{100-v} = \frac{-dt}{10}$$



$$\therefore \quad \ln(100-v) = -\frac{t}{10} + C_1.$$

$$\therefore \quad 100 - v = e^{(-t/10)+C_1} = e^{C_1}.e^{-t/10} = A.e^{-t/10}$$

where $A = e^{C_1}$. But $v(0)=0$. So $100-0 = A.e^0 = A$

$\therefore$ $A = 100$. $\therefore$ $100 - v = 100e^{-t/10} \Rightarrow v(t) = 100(1-e^{-t/10})$.

Since $e^{-t/10}$ is always positive, the velocity cannot reach (or exceed) $100\ ms^{-1}$. This is called the <u>limiting velocity</u> and can be found by finding

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 100(1-e^{-t/10}) = 100(1-0) = 100.$$

(b) We have $\frac{dz}{dt} = v(t) = 100(1-e^{-t/10})$.

So $dz = (100 - 100e^{-t/10})\,dt$

$$\therefore \quad z(t) = 100t - 100\left(\frac{-1}{1/10}\right)e^{-t/10} + C_2$$

$$= 100t + 1000e^{-t/10} + C_2$$

But $z(0)=0$, so $0 = 100(0) + 1000 + C_2 \Rightarrow C_2 = -1000$

$\therefore$ $z(t) = 100t + 1000e^{-t/10} - 1000 = 100(t + 10e^{-t/10} - 10)$

<u>Ex. 3</u> (Falling body problem with air resistance proportional to v^2)

A particle of mass 2kg falls from rest towards the earth from a great height. If the air resistance is λv^2 where $\lambda = \frac{1}{5}\ kg\,m^{-1}$, find

(a) the velocity of the particle, t seconds after its release

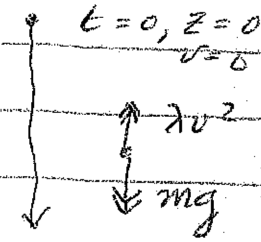
(b) the limiting velocity of the particle

(c) the distance the particle travels in t sec. after its release.

[Use $g = 10\ ms^{-2}$]

Sol. Let $z(t)$ = the distance the particle falls t sec. from release
Then $z(0) = 0$, $v = dz/dt$ and $v(0) = 0$.

(a) Now $\frac{d}{dt}(mv) = mg - \lambda v^2$



So $2\frac{dv}{dt} = 2(10) - \frac{1}{5}v^2$

$\therefore \; dv = \left(10 - \frac{v^2}{10}\right) = \frac{100 - v^2}{10}$

$\therefore \; \frac{20\,dv}{100 - v^2} = 2dt$

$\therefore \; \frac{20\,dv}{(10+v)(10-v)} = 2dt$

$\therefore \; \frac{dv}{10+v} + \frac{dv}{10-v} = 2dt$

$\therefore \; \ln(10+v) - \ln(10-v) = 2t + C_1$

Put $\frac{20}{(10-v)(10+v)} = \frac{A}{10+v} + \frac{B}{10-v}$
Then $20 = A(10-v) + B(10+v)$
Putting $v = -10$, gives us
$20 = A(10+10) + B(0) \Rightarrow A = 1$
Putting $v = 10$, gives us
$20 = A(0) + B(10+10) \Rightarrow B = 1$

$\therefore \; \ln\left(\frac{10+v}{10-v}\right) = 2t + C_1$. But $v(0) = 0$, so

$\ln\left(\frac{10+0}{10-0}\right) = 2(0) + C \Rightarrow C_1 = \ln(1) = 0$

$\therefore \; \ln\left(\frac{10+v}{10-v}\right) = 2t$. $\therefore \; \frac{10+v}{10-v} = e^{2t}$

$\therefore \; 10 + v = (10-v)e^{2t} \Rightarrow 10 + v = 10e^{2t} - ve^{2t}$

$\therefore \; v(e^{2t}+1) = 10(e^{2t}-1) \Rightarrow v = 10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$.

(b) limiting velocity $= \lim_{t\to\infty} v(t) = \lim_{t\to\infty} 10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$

$= \lim_{t\to\infty} 10\cdot\left(\frac{1-e^{-2t}}{1+e^{-2t}}\right) = 10\left(\frac{1-0}{1+0}\right) = 10 ms^{-1}$

(c) $\frac{dz}{dt} = v(t) = 10\cdot\left(\frac{e^{2t}-1}{e^{2t}+1}\right) = 10\cdot\left(\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right)$

$\therefore \; \int dz = 10\int\left(\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right)dt$

$\therefore \; z = 10\ln(e^{t}+e^{-t}) + C_2$

Put $u = e^{t}+e^{-t}$. Then
$du = (e^{t}-e^{-t})dt$
$10\int\cdots dt = 10\int\frac{du}{u} = 10\ln(u) + C_2$

But $z(0) = 0$. So

$0 = 10\ln(e^{0}+e^{-0}) + C_2 \Rightarrow C_2 = -10\ln(2)$

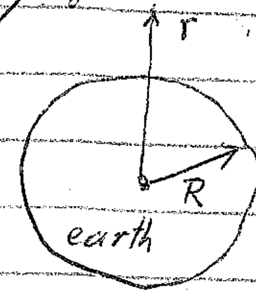
$\therefore \; z(t) = 10\ln(e^{t}+e^{-t}) - 10\ln(2) = 10\ln\left(\frac{e^{t}+e^{-t}}{2}\right)$.

Ex.4 (Escape-velocity from the earth's surface, no air resist.) (5)

A particle is thrown vertically upwards from the surface of the earth with great velocity V_0.

(a) Find the velocity of the particle when it at a distance r from the center of the earth?

(b) What is the smallest velocity needed to ensure that the particle never returns?



[Assume radius of earth, $R = 6371$ km and $g = 9.81 ms^{-2}$]

Sol. We know that according to the inverse square law the gravitation attraction of the earth is $-\frac{Km}{r^2}$ when the particle r units from the centre.

So $m\frac{dv}{dt} = -\frac{K}{r^2}m$. But when $r = R$, $m\frac{dv}{dt} = -g(m)$

$$\therefore \quad -\frac{K}{R^2}\cdot m = -g\cdot m \implies K = g\cdot R^2.$$

$$\therefore \quad m\frac{dv}{dt} = -\frac{g\cdot R^2}{r^2}\, m \implies \frac{dv}{dt} = -\frac{gR^2}{r^2}$$

(a) Now $\frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt}$ by the chain rule

$$= \frac{dv}{dr}\cdot v = v\frac{dv}{dr} \quad \text{bec. } v = \frac{dr}{dt}$$

$$\therefore \quad v\frac{dv}{dr} = -\frac{g\cdot R^2}{r^2} \implies 2v\,dv = 2gR^2\cdot\left(-\frac{dr}{r^2}\right)$$

$$\therefore \quad v^2 = 2gR^2\cdot\left(\frac{1}{r}\right) + C$$

But $v = V_0$ when $r = R$, so $V_0^2 = 2gR^2\cdot\frac{1}{R} + C$

$\therefore$ $C = V_0^2 - 2gR$. Hence

$$v^2 = \frac{2gR^2}{r} + (V_0^2 - 2gR)$$

So we can get v by taking square roots of both sides

(b) Since $\frac{2gR^2}{r}$ is always positive and tends to 0 as $r \to \infty$, we need $V_0^2 - 2gR$ to be ≥ 0 to ensure $v \geq 0$.

So if $V_0 \geq \sqrt{2gR} \simeq 11{,}180 ms^{-1}$ the particle will escape from earth. $\therefore$ escape-velocity $\simeq 11.180$ km/sec.

§2. Growth and decay problems

For many problems in the natural world, the rate of change of a quantity depends on the amount of the quantity present. If rate of change is directly proportional to the quantity, then we will have growth - if it is proportional to the negative of the quantity, the we will have decay. There are also more complicated situations in which we can have both growth or decay - depending on what amount of the quantity we started with.

<u>Radioactive decay</u>. The rate at which radioactive nuclei decay is directly proportional to the number of nuclei (and hence the mass) that are present in a given sample. So $\frac{dM}{dt} = -kM$ for radioactive substances. Now it will turn out that it takes the same amount of time for half of the mass M_0 of the substance to decay, no matter what amount M_0 we started out with. This time is called the <u>halflife</u> of the substance. Of course, different substances have different half-lives. The "k" is called the <u>decay-constant</u> of the substance.

<u>Ex. 1</u> (The relation between the decay-constant & the half-life)
A radioactive substance has decay constant k. Find the time it takes for half of the substance to decay.

<u>Sol.</u> Since k is the decay constant, $\frac{dM}{dt} = -kM$. Now let M_0 = mass of the substance we started out with at time $t=0$ and H = time it takes for half of it to decay. Then $\frac{dM}{M} = -k\,dt$. So $\ln(M) = -kt + C$

But $M(0) = M_0$. So $\ln(M_0) = -k(0) + C$.

$\therefore$ $C = \ln(M_0)$. So $\ln(M) = -kt + \ln(M_0)$

$\therefore$ $kt = \ln(M_0) - \ln(M) = \ln(M_0/M)$

$\therefore$ $t = \frac{1}{k} \ln\left(\frac{M_0}{M}\right)$. So if $M = \frac{1}{2}M_0$, then

$$t = \frac{1}{k} \cdot \ln\left(\frac{M_0}{M_0/2}\right) = \frac{1}{k}\ln\left(\frac{2M_0}{M_0}\right) = \frac{\ln 2}{k}.$$

Hence half-life $H = \frac{\ln(2)}{k}$. Also $k = \frac{\ln(2)}{H}$.

Ex 2. A certain radioactive isotope has a half-life of 3 seconds. If we start out with 4 grams of the isotope, how much of it will be there after 5 seconds?

Sol. We are given that $H = 3$. So $k = (\ln 2)/H = \frac{\ln(2)}{3}$

Now $\frac{dM}{dt} = -kM$. So $\frac{dM}{M} = -k\,dt$.

$\therefore$ $\ln(M) = -kt + C$. But $M(0) = 4$. So

$\ln(4) = -k(0) + C \Rightarrow C = \ln(4)$

$\therefore$ $\ln(M) = -kt + \ln(4)$.

$\therefore$ $M = e^{-kt + \ln(4)} = e^{\ln(4)} \cdot e^{-kt} = 4 \cdot e^{-kt}$

So $M(5) = 4 \cdot e^{-k \cdot 5} = 4e^{-(\ln 2) \cdot 5/3}$

$= 4 \cdot (e^{\ln 2})^{-5/3} = 2^2 \cdot (2)^{-5/3} = 2^{1/3}$ grams.

(Aside Note): When a radioactive substance decays, it changes into another substance – it doesn't just disappear. For example Carbon-14 is radioactive but Carbon-12 is not. When Carbon-14 decays, it becomes Nitrogen-14 and a small amount of energy is also radiated, in the form of electrons & photons. Because the percentage of C-14 is almost constant on earth and living things have the same matching percentage, we can check the percent of C-14 in long-dead life-forms and be able to tell when they died.

Newton's law of cooling & warming.

In addition to his laws of motion, Newton formulated many other laws. One of them is his law of cooling/heating. Let $\theta(t)$ be the temperature of a small object at time and $\theta_R(t)$ be the temperature of a big room in which the object is placed. Then

$$\frac{d\theta(t)}{dt} = -k\left[\theta(t) - \theta_R(t)\right] \quad \text{where } k \text{ is a positive constant.}$$

So if $\theta(t) > \theta_R(t)$ we'll have cooling of the object and if $\theta(t) < \theta_R(t)$ we'll have warming of the object. This law is reasonably accurate when $\theta(t) - \theta_R(t)$ is small & the heat transfer by radiation is also small. Usually $\theta_R(t)$ is assumed to be constant and the equation becomes $\frac{d\theta}{dt} = -k(\theta - \theta_R)$

Ex. 3 A body of temperature 32°C is placed in a room which has a constant temperature of 20°C. At the end of 4 minutes the temperature of the body is 24°C

(a) What will be the body's temperature after 6 minutes?

(b) How long will it take for the body to reach 21°C?

Sol. Let $\theta(t)$ be the temperature of the body after t min.

Then $\frac{d\theta}{dt} = -k(\theta - \theta_R) = -k(\theta - 20)$.

So $\frac{d\theta}{\theta - 20} = -k\,dt \quad \therefore \ln(\theta - 20) = -kt + C$

But $\theta(0) = 32$. So $\ln(32-20) = -k.0 + C$

$\therefore C = \ln 12$. Hence $\ln(\theta - 20) = -kt + \ln(12)$

$\therefore \ln(\theta - 20) - \ln(12) = -kt$

$\therefore \ln\left(\frac{\theta - 20}{12}\right) = -kt$. But $\theta(4) = 24$

$\therefore \ln\left(\frac{24-20}{12}\right) = -k.4 \Rightarrow \ln(4/12) = -4k$

So $\ln(1/3) = -4k \Rightarrow \ln 3 = 4k$.

Hence $k = (\ln 3)/4$. Thus

$$\ln\left(\frac{\theta - 20}{12}\right) = -\frac{\ln 3}{4}.t \quad (*)$$

(a) $\therefore (\theta - 20)/12 = e^{-\frac{\ln 3}{4}.t} = \left(e^{\ln 3}\right)^{-t/4} = (3)^{-t/4}$

$\therefore (\theta - 20) = 12.(3)^{-t/4} \Rightarrow \theta(t) = 20 + 12.3^{-t/4}$

$\therefore \theta(6) = 20 + 12.3^{-6/4} = 20 + 12.3^{-3/2}$

$= 20 + 4.3.3^{-3/2} = (20 + 4\sqrt{3}/3)\ ^\circ C$.

(b) If $\theta = 21$, then from equation $(*)$ we get

$$\ln\left(\frac{21-20}{12}\right) = -\frac{\ln 3}{4}.t$$

$\therefore \ln\left(\frac{1}{12}\right) = -\frac{\ln 3}{4}.t \Rightarrow -\ln(12) = -\frac{(\ln 3)}{4}.t$

$\therefore t = 4\ln(12)/\ln(3)$ minutes

Simple population growth

The laws of population growth varies according to the type of organisms with which we deal and also according to their enviroment. If the resources & space are unlimited, then a population of bacteria will grow according to the simple law

$$\frac{dP}{dt} = \lambda P \quad \text{where} \quad \text{is a positive constant}$$

and $P(t)$ is the population at time t. Here "λ" is the number of organisms a single bacteria will leave after 1 unit of time.

Ex.4 A colony of bacteria grows at a rate that is proportional to the number present. At time $t=0$, there are 4 million present and 2 hours later there are 12 million.

(a) How many bacteria will there be after 3 hours?

(b) How long will it take for the colony to reach 30 million?

Sol. Let $P(t)$ be the number of bacteria in millions at time t. (10)

Then $\frac{dP}{dt} = \lambda P$ and $P(0) = 4$ million.

So $\frac{dP}{P} = \lambda dt \Rightarrow \ln(P) = \lambda t + C$.

But $P(0) = 4$, so $\ln(4) = \lambda 0 + C \Rightarrow C = \ln(4)$

$\therefore \ln(P) = \lambda t + \ln 4 \Rightarrow \ln(P) - \ln(4) = \lambda t$

$\therefore \ln(P/4) = \lambda t$. Also $P(2) = 12$, so

$\ln(12/4) = \lambda . 2 \Rightarrow 2k = \ln(3) \Rightarrow \lambda = (\ln 3)/2$.

$\therefore \ln(P/4) = \frac{\ln 3}{2}.t$ (*)

From this we get $P/4 = e^{(\ln 3)t/2} \Rightarrow \frac{P}{4} = (e^{\ln 3})^{t/2} = (3)^{t/2}$

(a) $\therefore P(t) = 4.(3)^{t/2}$. Hence

$P(3) = 4.(3)^{3/2} = 4.3.\sqrt{3} = 12\sqrt{3}$ million.

(b) If $P = 30$, then from equation (*)

$\ln(30/4) = \frac{(\ln 3)}{2} t$

$\therefore t = \ln(15/2) / \frac{\ln 3}{2} = 2.\frac{\ln(15/2)}{\ln(3)}$ hours

Realistic population growth

Since resources and space are limited and organisms often die, the simple population growth law will only hold for microscopic organisms such as bacteria and will also only do so for fairly short periods of time. So if we let λ = no. of organisms that a single individual leaves behind after one unit of time, then λP will be the growth rate of P individuals. Also each individual must compete with P organisms for the finite resources, so each individual will have a casualty rate of μP — and hence the overall population will decrease by $(\mu P).P = \mu P^2$. Thus a more realistic law is $\frac{dP}{dt} = \lambda P - \mu P^2$

We usually write this as $\lambda P(1 - \frac{P}{K})$ where $K = \frac{\lambda}{\mu}$.

This constant K has a meaning. It will turn out to be the carrying-capacity of the environment where the organisms live.

Ex.5 The population of a certain colony of organisms satisfy the logistic law $\frac{dP}{dt} = \frac{1}{2} P\left(1 - \frac{P}{4000}\right)$, where t is measured in years. If $P(0) = 1000$,

(a) what will be the population after 2 years

(b) what will be the limiting population, i.e., the population $P(t)$ approaches as $t \to \infty$.

Sol. We have $\frac{dP}{dt} = \frac{1}{2} P\left(1 - \frac{P}{4000}\right) = \frac{1}{2} \cdot \frac{P(4000-P)}{4000}$

$\therefore \frac{4000\, dP}{P(4000-P)} = \frac{1}{2} dt.$

Put $\frac{4000}{P(4000-P)} = \frac{A}{P} + \frac{B}{4000-P}$

Then $4000 = A(4000-P) + B.P$

Putting $P=0$, gives $4000 = 4000A$

So $A = 1$. And putting $P = 4000$ gives $4000 = 4000B \Rightarrow B = 1$

$\therefore \left(\frac{1}{P} + \frac{1}{4000-P}\right) dP = \frac{1}{2} dt$

$\therefore \ln(P) - \ln(4000-P) = \frac{t}{2} + C$

$\therefore \ln\left(\frac{P}{4000-P}\right) = \frac{t}{2} + C.$

But $P(0) = 1000$, so $\ln\left(\frac{1000}{4000-1000}\right) = \frac{0}{2} + C$

$\therefore C = \ln(\frac{1}{3})$ Hence

$$\ln\left(\frac{P}{4000-P}\right) = \frac{t}{2} + \ln(1/3)$$

$\therefore \frac{P}{4000-P} = e^{t/2 + \ln(1/3)} = e^{\ln(1/3)} \cdot e^{t/2} = \frac{1}{3} e^{t/2}$

$\therefore \frac{4000-P}{P} = \frac{3}{e^{t/2}} = 3e^{-t/2}$

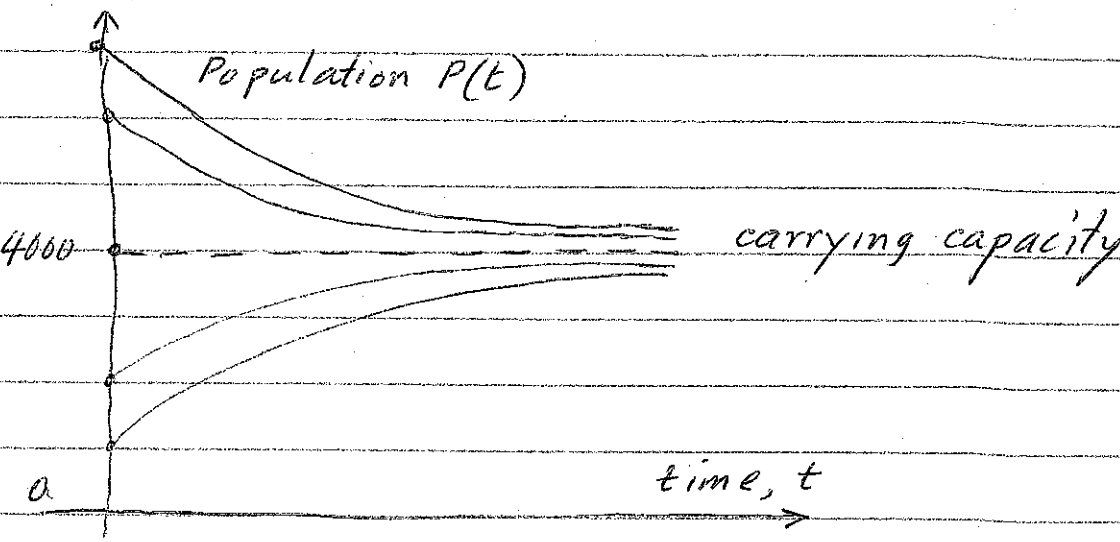
$\therefore \frac{4000}{P} - 1 = 3e^{-t/2} \Rightarrow 1 + 3e^{-t/2} = \frac{4000}{P}$

(a) $\therefore P(t) = 4000/(1+3e^{-t/2})$. So after 2 years the population will be $P(2) = 4000/(1+3e^{-1})$

(b) The limiting population will be

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{4000}{1+3e^{-t/2}} = \frac{4000}{1+0} = 4000$$

Note: If we start with a population $0 < P(0) < 4000$, then $P(t)$ will approach 4000 from below as $t \to \infty$. If we start with a population $P(0) > 4000$ then $P(t)$ will approach 4000 from above as $t \to \infty$. And if $P(0) = 4000$, then $P(t) = 4000$ for all t.



This was why in the equation $\frac{dP}{dt} = \frac{1}{2}P\left(1 - \frac{P}{4000}\right)$ the 4000 is called the carrying capacity, since this is the number at which the population will stabilize.

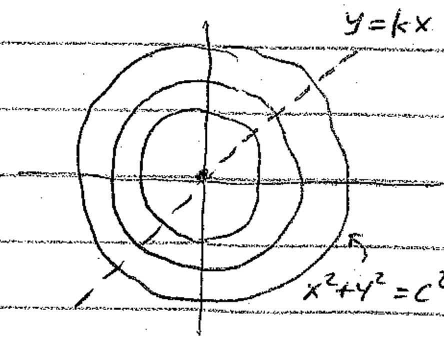
Unfortunately, organisms – such a fish are not left alone. Human beings (and other animals) harvest them. If only beings are doing the harvesting at a constant rate H per unit time, the realistic growth ODE becomes $\frac{dP}{dt} = \lambda P\left(1 - \frac{P}{K}\right) - H$. This equation can be solved and points the way to managing our finite resources. If other animals prey on the organisms, then we get coupled O.D.E s.

§3. Orthogonal trajectories

<u>Def</u> Let $F(x,y,c)=0$ be a given one-parameter family of curves in the xy plane. A curve that intersects each of the curves in the given family is called an <u>orthogonal trajector</u> of that given family.

Orthogonal trajectories are used as curvilinear coordinates in Pure Mathematics — and in Physics they are used to represent force fields (such as the electric field) and their equi-potentials

<u>Ex.1</u> Consider the family of curves $x^2+y^2=c^2$. This is a family of circles of radius c with centers at the origin. Any straight line, $y=kx$, through the origin is an orthogonal trajectory for the given family



<u>Algorithm</u> for finding all the orthogonal trajectories of a given family.

Step. 1: From the equation $F(x,y,c)=0$, differentiate w.r.t. x and then eliminate c from the two equations $F(x,y,c)=0$ & $F'(x,y,c)=0$ to get the ODE $\frac{dy}{dx}=f(x,y)$ of the family.

Step. 2 Form the associated ODE $\frac{dy}{dx}=\frac{-1}{f(x,y)}$ and solve this ODE to get the general solution $G(x,y,c)=0$.

<u>Step 3</u> The family $G(x,y,c)=0$ will be the set of all orthogonal trajectories of the family $F(x,y,c)=0$. Check that each member is orthogonal to the second family.

Ex.2 Find all the orthogonal trajectories of the one-parameter family of curves $y - cx = 0$

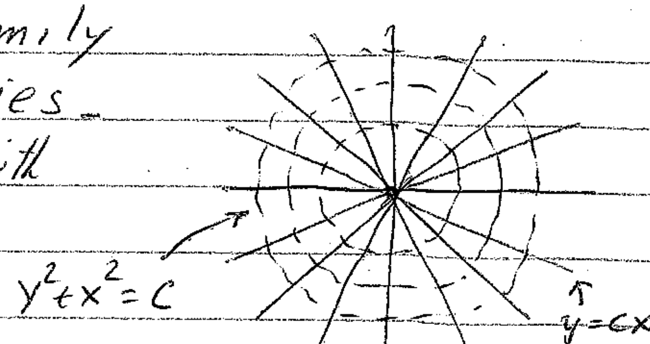
Sol. We have $y - cx = 0$. Differentiating w.r.t. x gives us $\frac{dy}{dx} - c = 0$. But from $y - cx = 0$, we know that $c = y/x$. Hence $\frac{dy}{dx} = c = \frac{y}{x}$

So the ODE for the given family is $\frac{dy}{dx} = \frac{y}{x}$

The ODE for the family of orthogonal trajectories is $\frac{dy}{dx} = -1/\left(\frac{y}{x}\right) = \frac{-x}{y}$ $\therefore\ y\,dy = -x\,dx$

$\therefore\ 2y\,dy + 2x\,dx = 0$. $\therefore\ y^2 + x^2 = C$

and this is our family of orthogonal trajectories - a family of circles with center at the origin.



Ex.3 Find all the orthogonal trajectories of the family of curves given by $y - cx^2 = 0$

Sol. We have $y - cx^2 = 0$. So $dy/dx - 2cx = 0$.

But $c = y/x^2$, so $dy/dx - 2(y/x^2) \cdot x = 0$

$\therefore$ ODE of given family is $\frac{dy}{dx} = \frac{2y}{x}$

The ODE of the family of orthogonal trajectories is $\frac{dy}{dx} = -1/\left(\frac{2y}{x}\right) = \frac{-x}{2y}$

$\therefore\ 2y\,dy = -x\,dx$

$\therefore\ 2y\,dy + x\,dx = 0$

$\therefore\ y^2 + \frac{x^2}{2} = C_1 = C^2$

$\therefore\ \frac{y^2}{C^2} + \frac{x^2}{(C\sqrt{2})^2} = 1$

This is our family of orthogonal trajectories - ellipses with $\sqrt{2}:1$ ratios