§1. Linear constant coeff. homogeneous ODEs (Real roots)

In this chapter we will be mostly concerned with linear constant-coefficient ODEs. We have already seen how to solve any linear first-order ODEs by using integrating factors. So we will pay more attention to linear ODEs which are of order 2 or higher. The methods, of course, will apply to first-order linear ODEs. First let us consider linear constant-coefficient homogeneous ODEs.

Def. Recall that a linear ODE of order n is any ODE of the form $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$. This ODE is said to be homogeneous if $b(x) \equiv 0$. (Homogeneous has another meaning from Ch. 1 – but this can't be helped because historically both kinds of ODEs got the names because of different reasons.)

Def. A linear constant-coefficient ODE of order n is one of the form $y^{(n)} + a_{n-1}\cdot y^{(n-1)} + \cdots + a_1\cdot y' + a_0 y = b(x)$ where $a_0, a_1, a_2, \ldots, a_{n-1}$ are fixed constants. The $b(x)$ can be any function of x.

Let us try the simplest kind of higher order ODEs – the linear constant-coefficient homogeneous ODEs.

Ex. 1 Find the general solution of the ODE $y'' - 5y' + 6y = 0$. (*)

Sol. Suppose $y = e^{\alpha x}$ is a solution of (*). Then $y' = \alpha\cdot e^{\alpha x}$ and $y'' = \alpha^2 e^{\alpha x}$. So (*) becomes

Ex. 1

$$\alpha^2 . e^{\alpha x} - 5\alpha . e^{\alpha x} + 6e^{\alpha x} = 0$$

$\therefore\ (\alpha^2 - 5\alpha + 6) . e^{\alpha x} = 0$

Since $e^{\alpha x}$ is never 0, $\alpha^2 - 5\alpha + 6 = 0$

$\therefore\ (\alpha - 2)(\alpha - 3) = 0 \implies \alpha = 2$ or 3.

So $y = e^{2x}$ and $y = e^{3x}$ should both be solutions of (*). A quick check shows that this is indeed so.

Now the theory of Homogeneous linear ODEs tells us that if we have n linearly independent solutions then the general solution will be the collection of all linear combinations of these n independent solutions.

Since (*) is a homogeneous ODE of order 2, the general solution will be $y = C_1 e^{2x} + C_2 e^{3x}$, where C_1 & C_2 are arbitrary constants.

Ex. 2 Find the general solution of the ODE

$$y'' - 6y' + 9y = 0 \qquad (*)$$

Sol Again we suppose that $y = e^{\alpha x}$ is a solution. Then $y' = \alpha . e^{\alpha x}$ and $y'' = \alpha^2 . e^{\alpha x}$. So (*) becomes

$$\alpha^2 e^{\alpha x} - 6\alpha . e^{\alpha x} + 9 . e^{\alpha x} = 0 \implies (\alpha^2 - 6\alpha + 9) e^{\alpha x} = 0$$

$$\implies \alpha^2 - 6\alpha + 9 = 0 \ (\text{bec. } e^{\alpha x} \neq 0) \implies (\alpha - 3)^2 = 0.$$

So $\alpha = 3$ (twice). So $y = e^{3x}$ will be a solution of (*). But we need another linearly independent solution in order to get the general solution.

Let us put $v = y' - 3y$. Then (*) becomes

$$y'' - 3y' - 3y' + 9y = 0$$

$$\therefore\ (y'' - 3y') - 3(y' - 3y) = 0$$

$$\therefore\ (y' - 3y)' - 3(y' - 3y) = 0$$

$\therefore\ v' - 3v = 0$. Now this is a first-order, linear homogeneous ODE with auxiliary equation $\alpha - 3 = 0$.

Ex.2 So $\alpha = 3$. Hence $v = e^{3x}$ is a solution of the ODE $v' - 3v = 0$. And so the general solution of $v' - 3v = 0$ will be $v = C_2 e^{3x}$.

But $v = y' - 3y$. Hence $y' - 3y = v = C_2 e^{3x}$. Now this is a linear first-order ODE and we can solve for y.

Integrating factor $= e^{\int -3dx} = e^{-3x}$. Hence

$$e^{-3x} . y' - 3 . e^{-3x} . y = C_2 . e^{3x} . e^{-3x}$$

Thus $\frac{d}{dx}(ye^{-3x}) = C_2 \Rightarrow y . e^{-3x} = C_2 x + C_1$

$$\therefore \quad y = (C_2 x + C_1) e^{3x} \quad \ldots \quad (**)$$

By taking $C_2 = 1$ and $C_1 = 0$, we see that $y = x . e^{3x}$ is a linearly independent solution from $y = e^{3x}$. Hence the general solution will be $y = C_1 . e^{3x} + C_2 . xe^{3x}$.

[This is what we also got in (**)]

Let us now state the relevant theorem which tells us how to find the general solution — and also say what it means for a set of solutions to be linearly independent.

<u>Def.</u> Let $\{f_1, f_2, \ldots, f_k\}$ be a set of functions with domain (a, b). We say that $\{f_1, f_2, \ldots, f_k\}$ is <u>linearly dependent</u> if we can find constants $c_1, \ldots, c_k$; not all zeros, such that $c_1 . f_1(x) + c_2 . f_2(x) + \ldots + f_k(x) = 0$ for all x in (a,b).

The set $\{f_1, f_2, \ldots, f_k\}$ is linearly <u>independent</u> if it is not linearly <u>dependent</u>. This is <u>logically equivalent</u> to saying $c_1 . f_1(x) + c_2 f_2(x) + \cdots + c_k . f_k(x) = 0$ for all x in (a,b)

$$\Rightarrow \quad c_1 = c_2 = \cdots = c_k = 0.$$

<u>Ex.3</u> Let $f_1(x) = x$, $f_2(x) = x + 1$, and $f_3(x) = x - 1$ for $x \in (-\infty, \infty)$

Which of the following sets are linearly independent?

Ex.3 (a) $\{f_1, f_2\}$ (b) $\{f_2, f_3\}$ (c) $\{f_1, f_2, f_3\}$

Sol: (a) Suppose $c_1 . f_1(x) + c_2 . f_2(x) \equiv 0$ [i.e., it is $= 0$ for all $x \in (-\infty, \infty)$]

Then $c_1 . x + c_2 . (x+1) \equiv 0$

$\therefore (c_1 + c_2)x + c_2 \equiv 0 \equiv 0.x + 0$

$\therefore c_1 + c_2 = 0$ (equating the coeff. of x)

and $c_2 = 0$ (equating the constant terms)

$\therefore c_2 = 0$ and $c_1 = 0 - c_2 = 0$. Hence $\{f_1, f_2\}$ is linearly independent.

(b) Suppose $c_1 . f_2(x) + c_2 . f_3(x) \equiv 0$. Then

$$c_1 . (x+1) + c_2 . (x-1) \equiv 0$$

$\therefore (c_1 + c_2)x + (c_1 - c_2) \equiv 0$

$\therefore c_1 + c_2 = 0$ and $c_1 - c_2 = 0$

$\therefore c_2 = -c_1$ and so $c_1 - (-c_1) = 0 \Rightarrow 2c_1 = 0$

$\therefore c_1 = 0$ and $c_2 = -c_1 = 0$.

Hence $\{f_2, f_3\}$ is linearly independent.

(c) Suppose $c_1 . f_1(x) + c_2 . f_2(x) + c_3 . f_3(x) \equiv 0$.

Then $c_1 . x + c_2 (x+1) + c_3 . (x-1) \equiv 0$

$\therefore (c_1 + c_2 + c_3)x + (c_2 - c_3) \equiv 0$

$\therefore c_2 - c_3 = 0$ and $c_1 + c_2 + c_3 = 0$

$\therefore c_2 = c_3$ and $c_1 = -c_2 - c_3 = -2c_2$

So by taking $c_2 = 1$, $c_3 = 1$ and $c_1 = -2$, we will get a non-trivial solution of $c_1 . f_1(x) + c_2 f_2(x) + c_3 f_3(x) \equiv 0$

Hence $\{f_1, f_2, f_3\}$ is not linearly dependent.

Now this all looks like tedious work and it would be nice if there is a simpler, faster way to tell if a set of functions on (a,b) is linearly independent. And indeed there is a simpler way using the Wronskian of the set of functions. Also our previous method seems to work well only for polynomials.

Def. Let $\langle f_1, f_2, \ldots, f_k \rangle$ be a sequence of k functions which are $(k-1)$ times differentiable on (a,b). We define the Wronskian of $\langle f_1, \ldots, f_k \rangle$ by

$$W(f_1, f_2, \ldots, f_k) = \begin{vmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ \vdots & & & \\ f_1^{(k-1)} & f_2^{(k-1)} & \cdots & f_k^{(k-1)} \end{vmatrix}$$

Theorem 1 (The Wronskian all-or-none theorem)
If $f_1, \ldots, f_n$ are n solutions of homog. linear n-th order ODE and $W(f_1, \ldots, f_n) \not\equiv 0$ on (a,b), then $\{f_1, \ldots, f_n\}$ is lin. indep.

Ex. 4 Show that (a) $\{e^{2x}, e^{3x}\}$ (b) $\{e^x, xe^x\}$ are linearly independent solutions of the ODEs $y'' - 5y' + 6y = 0$ and $y'' - 2y' + y = 0$, respectively

Sol. (a) We know that e^{2x} & e^{3x} are solutions of $y'' - 5y' + 6y = 0$.
Now $W(e^{2x}, e^{3x}) = \begin{vmatrix} e^{2x} & e^{3x} \\ (e^{2x})' & (e^{3x})' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

$= e^{2x} \cdot 3e^{3x} - 2e^{2x} \cdot e^{3x} = e^{5x} \not\equiv 0$.

So from Theorem 1, it follows that $\{e^{2x}, e^{3x}\}$ is linearly indep.

(b) The auxiliary equation of $y'' - 2y' + y = 0$ is $\alpha^2 - 2\alpha - 1 = 0$.
So $(\alpha - 1)^2 = 0$. $\therefore \alpha = 1$ (twice). From this we can deduce as in Ex. 2 that $y = e^x$ and $y = xe^x$ are solutions of $y'' - 2y' + y = 0$. Now

$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ (e^x)' & (xe^x)' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1) \cdot e^x \end{vmatrix}$

$= e^x \cdot (x+1) \cdot e^x - e^x \cdot x \cdot e^x = e^{2x} \not\equiv 0$

So from Theorem 1, it follows that $\{e^x, xe^x\}$ is linearly indep.

Theorem 2 (General solution for linear homog. ODEs)
Suppose $f_1(x), f_2(x), \dots, f_n(x)$ are n linearly independent solutions of the linear homogeneous ODE

$$y^{(n)} + a_{n-1}(x) \cdot y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) \equiv 0 . \quad (*)$$

Then $C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n \cdot f_n(x)$ is the gen. solution of $(*)$.

We will use D (or D_x when there are more than one independent variables) to denote the derivative w.r.t. x. Then $\frac{dy}{dx}$ becomes Dy, $\frac{d^2y}{dx^2}$ becomes $D^2 y$, and so on. And any linear constant-coefficient ODE can then be written in the form $\mathcal{L}(D) y = b(x)$ where $\mathcal{L}(D)$ is a polynomial in D.

Ex. 6 Find the general solution of the ODE

$$y''' - y'' - y' + y = 0 \qquad (*)$$

Sol. If we use $D = \frac{d}{dx}$, then $(*)$ becomes

$$(D^3 - D^2 - D + 1) y = 0 .$$

So the auxiliary equation is $D^3 - D^2 - D + 1 = 0$.
Now if $\mathcal{L}(D) = D^3 - D^2 - D + 1$, then $\mathcal{L}(1) = 0$. So $(D-1)$ is a factor of $\mathcal{L}(D)$. So

$$(D-1)(D^2 + aD - 1) = D^3 - D^2 - D + 1$$

So $-D^2 + aD^2 = -D^2$ and $-aD - D = -D$. So $a = 0$.

$$\therefore \quad \mathcal{L}(D) = (D-1)(D^2-1) = (D-1)(D-1)(D+1)$$
$$= (D-1)^2 (D+1)$$

$\therefore$ $D = 1$ (twice) or $D = -1$. So $y = e^x$, $y = xe^x$ and $y = e^{-x}$ are three linearly independent solutions (Check this by using the Wronskian, if you are not sure.)
So the general solution by Theorem 2 will be

$$y = C_1 e^x + C_2 x e^x + C_3 e^{-x} .$$

§2. Linear Constant-Coefficient homogeneous ODEs (complex roots)

Ex. 1 Find the general solution of the ODE $y'' + y = 0 \quad \ldots (*)$

Sol. By letting $D = \frac{d}{dx}$, we can write $(*)$ as $(D^2+1)y = 0$. So the auxiliary equation is $D^2+1 = 0 \Rightarrow (D-i)(D+i) = 0$. $\therefore D = i$ or $-i$. So $y = e^{ix}$ and $y = e^{-ix}$ are two linearly independent solutions. Hence the general solution of $(*)$ will be $y = C_1 e^{ix} + C_2 e^{-ix}$. End of story! No so fast! But wait, what is e^{ix}? Well, just look at what e^x is.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\therefore e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots + \frac{(ix)^{2n}}{(2n)!} + \frac{(ix)^{2n+1}}{(2n+1)!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$+ i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= \cos(x) + i\sin(x).$$ But this is a complex solution.

Also $e^{-ix} = e^{i(-x)} = \cos(-x) + i\sin(-x) = \cos x - i\sin(x)$.

So $y = \cos(x) + i\sin x$ & $y = \cos(x) - i\sin(x)$ are solutions of $(*)$. Taking $C_1 = \frac{1}{2}$ & $C_2 = \frac{1}{2}$ gives us

$$y = \tfrac{1}{2}(\cos(x) + i\sin x) + \tfrac{1}{2}(\cos x - i\sin x) = \cos(x)$$ is a sol.

And taking $C_1 = -i/2$ and $C_2 = i/2$, gives us

$$y = \frac{-i}{2}(\cos x + i\sin x) + \frac{i}{2}(\cos x - i\sin x) = \sin x$$ is a sol.

Also $\begin{vmatrix} \cos x & \sin x \\ (\cos x)' & (\sin x)' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$.

Hence $y = \cos x$ & $y = \sin x$ are two lin. indep. real solutions. So the general solution of $(*)$ is $y = A\cos x + B\sin x$ where A and B are arbitrary real constants.

Ex. 2 Find the general solution of the 4th-order ODE
$$(D^2-2D+5)^2 y = 0.$$

Sol. We instantly get that the auxiliary equation is $(D^2-2D+5)^2 = 0$. So $D = \left[-(-2) \pm \sqrt{4-4(5)}\right]/2$ (twice)
$= 1 \pm 2i$ (twice).

$\therefore$ four linearly independent solutions are
$y = e^{(1+2i)x}$, $y = x.e^{(1+2i)x}$, $y = e^{(1-2i)x}$, & $y = x.e^{(1-2i)x}$.
Now $e^{(1+2i)x}$ & $e^{(1-2i)x}$ produces $e^x\cos(2x)$ & $e^x\sin(2x)$.
And $xe^{(1+2i)x}$ & $x.e^{(1-2i)x}$ produces $xe^x\cos(2x)$ & $x.e^x\sin(2x)$.
So $y = A_1 e^x\cos(2x) + A_2 x e^x\cos(2x) + B_1 e^x \sin(2x) + B_2 x e^x \sin(2x)$.
$= (A_1 + A_2 x)e^x\cos(2x) + (B_1 + B_2 x)e^x \sin(2x)$.

We can summarise the results of all the work in the examples we did so far in the following theorems.

Theorem 3A (Distinct real roots theorem)
Let $\mathcal{L}(D)y = 0$ be a homog. linear constant-coeff. nth order ODE. If the roots of the auxiliary equation; $\alpha_1, \dots, \alpha_n$ are all real and distinct, then the general solution of $\mathcal{L}(D)y=0$ is $y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \dots + c_n . e^{\alpha_n x}$.

Theorem 3B (Repeated root with multiplicity m)
If the root α, of the aux. eq. $\mathcal{L}(D) = 0$, is of multiplicity m, then this leads to exactly m linearly independent solutions
$y_1 = e^{\alpha x}$, $y_2 = xe^{\alpha x}$, ..., $y_m = x^{m-1}.e^{\alpha x}$.

Theorem 3C (Complex root with multiplicity m)
If the root $a+ib$, of the aux. eq., is of multiplicity m; then $a-ib$ is also a root, of the aux. eq, of multiplicity m — and these two roots lead to exactly $2m$ linearly independent solutions

$y_1 = e^{ax}.\cos(bx)$, $y_2 = x.e^{ax}.\cos(bx)$, $\dots$, $y_m = x^{m-1}.e^{ax}.\cos(bx)$
$y_{m+1} = e^{ax}.\sin(bx)$, $y_{m+2} = x.e^{ax}\sin(bx)$, $\dots$, $y_{2m} = x^{m-1}.e^{ax}.\cos(bx)$.

Theorem 4 (Existence & Uniqueness theorem for linear ODEs)
Suppose that in the n-th order linear ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x).y' + a_0(x).y = b(x) \dots (*)$$

the coefficients $a_0(x), a_1(x), \dots, a_{n-1}(x)$ and the RHS term $b(x)$ are all continuous on some interval (a,b) and $x_0 \in (a,b)$. Suppose also that $c_0, c_1, c_2, \dots, c_{n-1}$ are n chosen constants. Then there exists a unique solution $y = f(x)$ of $(*)$ with $f(x_0) = c_0$, $f'(x_0) = c_1$, $\dots$, and $f^{(n-1)}(x_0) = c_{n-1}$.

We shall give an example of Theorem 4 when $b(x) \equiv 0$ because so far we are only able to solve homogeneous ODEs. Later on we will deal with $b(x) \not\equiv 0$.

Ex. 3 Find the unique solution of the ODE $y'' - y' - 2y = 0$ with the initial conditions $y(0) = 7$ and $y'(0) = 2$

Sol. We have $(D^2 + D - 2)y = 0$. So the aux. eq. is
$D^2 - D - 2 = 0$. $\therefore (D+1)(D-2) = 0$
$\therefore D = 2$ or -1. $\therefore y = A.e^{2x} + B.e^{-x}$
So $y' = 2A.e^{2x} - B.e^{-x}$
But $y(0) = 7$, so $7 = A.e^0 + B.e^{-0}$
And $y'(0) = 2$, so $2 = 2A.e^0 - B.e^{-0}$
$\therefore A + B = 7$, so $B = 7 - A$
$2A - B = 2$, so $2A - (7 - A) = 2$
$\therefore 3A = 9 \Rightarrow A = 3$. So $B = 7 - A = 7 - 3 = 4$.
Hence the unique solution with $y(0) = 7$ & $y'(0) = 2$ is $y = 3e^{2x} + 4e^{-x}$.

§3. Non-homogeneous linear constant-coefficient ODEs

In the first two sections of this chapter we indicated how to solve any homogeneous linear const.-coeff ODE. Unfortunately in the corresponding non-homogeneous ODE, the RHS $b(x)$ can be any function of x — and this makes it difficult (and sometimes impossible) to solve the ODE in closed form. When the RHS $b(x) = q(x).e^{\alpha x}$ where $q(x)$ is a polynomial in x, the non-homogeneous ODE is reasonably easy to solve. First we have a general theorem for all linear ODEs

Theorem 4 (Non-homogeneous linear ODE theorem)
Let y_p be a particular solution of the n-th order linear ODE

$$y^{(n)} + a_{n-1}(x).y^{(n-1)} + \cdots + a_1(x)\,y' + a_0(x).y = b(x) \quad \ldots(**)$$

and y_c be the general solution of the corresponding homog. ODE

$$y^{(n)} + a_{n-1}(x).y^{(n-1)} + \cdots + a_1(x).y' + a_0(x).y = 0 \quad \ldots(*)$$

Then the general solution of $(**)$ is $y = y_c + y_p$.

Ex.1 Find the general solution of the ODE $y'' - 4y = 20e^{3x}$, $(**)$

Sol. First let us find y_c, the general solution of the corresponding homog. ODE. We have $y'' - 4y = 0$

$\therefore (D^2 - 4)y = 0 \qquad \therefore (D-2)(D+2)y = 0$

$\therefore D = 2$ or -2. So $y_c = C_1 e^{2x} + C_2 e^{-2x}$.

Now since the RHS$(**)$ is e^{x}, let us try

$y_p = a.e^{3x}$. Then $y_p' = 3ae^{3x}$ and $y_p'' = 9ae^{3x}$.

So $(**)$, which is $y_p'' - 4y_p = 20e^{3x}$, becomes

$$9a.e^{3x} - 4ae^{3x} = 20e^{3x}$$

$$\therefore 5ae^{3x} = 20.e^{3x} \Rightarrow a = 4$$

$\therefore y_p = 2e^{3x}$. So the general solution of $(**)$ will be $y = y_c + y_p = C_1 e^{2x} + C_2 e^{-2x} + 4e^{3x}$.

Ex.2 Find the general solution of the ODE

$$(D^2+D-6)y = 15e^{2x} \quad \text{---} (**)$$

Sol. We have $D^2+D-6=0$ as the auxiliary equation.
So $(D-2)(D+3)=0$. $\therefore D=2$ or $D=-3$. Hence the general solution of the corresponding homog. eq. $(D^2+D-6)y=0$ is $y_c = C_1e^{2x} + C_2e^{-3x}$ (this is called the complementary sol.)
Since the RHS(**) is $15e^{2x}$, it appears that we should try $y_p = a.e^{2x}$. So $y_p' = 2a.e^{2x}$ & $y_p'' = 4a.e^{2x}$.
So $(D^2+D-6)y = 15e^{2x}$ becomes $y_p''+y_p'-6y_p = 15.e^{2x}$

$\therefore \quad 4a.e^{2x} + 2a.e^{2x} - 6a.e^{2x} = 15.e^{2x}$

$\therefore \quad (4a+2a-6a).e^{2x} = 15.e^{2x}$.

$\therefore \quad 0 = 15$. So something went wrong!

Well, we got $0=15$, because we assumed that there was a particular solution of the form $y_p = a.e^{2x}$. The truth of the matter is that there can be no particular solution of this form because $a.e^{2x}$ is a part of y_c.

So let us try $y_p = ax.e^{2x}$. Then

$$y_p' = (a+2ax).e^{2x}$$

and $$y_p'' = [2a+2(a+2ax)].e^{2x} = (4a+4ax)e^{2x}$$

So $y_p''+y_p'-6y_p = 15.e^{2x}$ becomes

$$(4a+4ax)e^{2x} + (a+2ax)e^{2x} - 6.ax.e^{2x} = 15e^{2x}$$

$\therefore \quad (4ax+2ax-6ax).e^{2x} + (4a+a).e^{2x} = 15e^{2x}$

$\therefore \quad 5a.e^{2x} = 15e^{2x} \Rightarrow 5a = 15 \Rightarrow a=3$.

$\therefore \quad y_p = 3x.e^{2x}$ (not $3.e^{2x}$)

Hence the general solution of (**) becomes

$$y = y_c + y_p \quad \text{(complementary \& particular solutions)}$$
$$= C_1.e^{2x} + C_2e^{-3x} + 3x.e^{2x}.$$

Ex.3 Find the general solution of the linear ODE

$$y'' + y' - 6y = 12x \qquad (**)$$

Sol. The corresponding homog. ODE is $y''+y'-6y=0$.
So $(D^2+D-6)y=0$. Aux. eq. is $D^2+D-6=0$
$\therefore$ $(D+3)(D-2)=0$. $\therefore$ $D = 2$ or -3
So $y_c = C_1 e^{2x} + C_2 e^{-3x}$.
Since the RHS$(**)$ is $12x$, it seems reasonable to try $y_p = ax$ since $12x = 12x . e^{0x}$ and 0 is not a root of the aux. eq. So let's try $y_p = ax$.
Then $y_p' = a$ and $y_p'' = 0$. So $(**)$ becomes

$$0 + a - 6.ax = 12x.$$

$\therefore$ $a - 6a.x = 0 + 12x$

$\therefore$ $a = 0$ and $-6a = 12$

So $a=0$ and $a=-2$. But this is impossible!
Something went wrong – but what?

The thing that went wrong was that the y_p we tried was not "closed" under the differential operator $\mathcal{L}(D) = D^2+D-6$. If we had tried a y_p of the form $y_p = ax+b$ we would not have obtained any surprise outside terms in the LHS$(**)$.

So put $y_p = ax+b$. Then $y_p' = a$ & $y_p'' = 0$. So $(**)$ becomes $y_p'' + y_p' - 6y_p = 0$

$\therefore$ $0 + a - 6(ax+b) = 12.x$

$\therefore$ $(a-6b) - 6ax = 0 + 12.x \Rightarrow -6a=12$ & $a-6b=0$

$\therefore$ $a=-2$ and so $-2-6b=0 \Rightarrow b = -1/3$.

$\therefore$ $y_p = -2x - 1/3$. Hence the general solution will be

$$y = y_c + y_p = C_1 e^{2x} + C_2 e^{-2x} - 2x - 1/3.$$

Ex. 4 Find the general solution of the ODE

$$y'' - 3y' + 2y = 20\sin x \qquad (**)$$

Sol. The corresponding homog. ODE is $y'' - 3y' + 2y = 0$

So the aux. equation is $D^2 - 3D + 2 = 0$

$\therefore (D-1)(D-2) = 0 \Rightarrow D = 1$ or 2. $\therefore y_c = C_1 e^x + C_2 e^{2x}$.

Now the RHS(**) has the term $20\sin x$ in it and since it is not "closed" under the differential operator $\mathcal{L}(D) = D^2 - 3D + 2$, we should not try $y_p = b.\sin x$ because of our experience from Ex. 3.

So let us try $y_p = a.\cos x + b.\sin x$. Why? Well that is closed under the operator D, and so it is automatically closed under the operator $\mathcal{L}(D)$. So

$$y_p' = -a\sin x + b\cos x$$

$$y_p'' = -a\cos x - b\sin x$$

$\therefore$ (**) becomes $y_p'' - 3y_p' + 2y_p = 20\sin x$ which is

$(-a\cos x - b\sin x) - 3(-a\sin x + b\cos x) + 2(a\cos x + b\sin x) = 20\sin x$

$\therefore (-a - 3b + 2a)\cos x + (-b + 3a + 2b)\sin x = 0.\cos x + 20\sin x$

$\therefore \quad -3b + a = 0 \quad \Rightarrow a = 3b$

and $\quad 3a + b = 20 \quad \Rightarrow 9b + b = 20 \Rightarrow b = 2$

$\therefore \quad a = 3b = 6$. Hence $y_p = 6\cos x + 2\sin x$

So general solution is given by

$$y = y_c + y_p = C_1 e^x + C_2 e^{2x} + 6\cos x + 2\sin x$$

From our experience in Ex. 1–4, we can see why the following theorems will be what we ought to expect.

Theorem 5A (Method of undetermined coefficients, α real)

If $q(x)$ is a polynomial of degree k and α is real root of $\mathcal{L}(D) = 0$, of multiplicity m, then the minimal form of a particular solution of the ODE $\mathcal{L}(D)y = q(x).e^{\alpha x}$

will be $y_p = (a_0 + a_1 x + a_2 x^2 + \cdots + a_k) \cdot x^m \cdot e^{\alpha x}$

Theorem 5B (Method of undetermined coefficients, α complex)
If $q(x)$ is a polynomial of degree k and $\alpha = a + ib$ is a complex root, of the auxiliary equation $\mathcal{L}(D) = 0$, of multiplicity m, then the minimal form of a particular solution of the ODE $\mathcal{L}(D) = q(x) \cdot \{e^{ax} \cdot \cos(bx)\}$ or $\{q(x) \cdot e^{ax} \cdot \sin(bx)\}$, will be

$$y_p = (a_0 + a_1 x + \cdots + a_k x^k) \cdot x^m \cdot e^{ax} \cdot \cos(bx) + (b_0 + b_1 x + \cdots + b_k x^k) \cdot x^m \cdot e^{ax} \cdot \sin(bx)$$

Ex. 5 Find the complementary solution of each of the following ODE and give the minimal form of a particular solution to each of them.

(a) $D^2 (D-1)(D+2)^2 y = x^3 + 4$.

(b) $(D^2 - 1)^2 (D^2 + 1) y = 6x^2 + e^x$.

(c) $(D+3)^2 (D-2)^3 y = 5x \cdot e^{2x}$.

(d) $(D^2+1)^2 (D+1)^3 = 6x \sin x$.

(e) $(D^2 - 2D + 5)^2 (D^2 + 1) = x e^x \cdot \cos(2x)$.

Sol. (a) We have $(D-0)^2 (D-1)(D+2)^2 y = 0$ as the homog. ODE. So $D = 0$ (twice), 1 (once), and -2 (twice)

$\therefore \; y_c = (C_1 + C_2 x) e^{0x} + C_3 e^x + (C_4 + C_5 x) e^{-2x}$

Notice this is a 5-th order ODE, so we should have 5 linearly independent solutions and hence 5 arb. constants. Since $x^3 + 4 = (x^3 + 4) \cdot e^{0x}$ and $x^3 + 4$ is a polynomial of degree 3 and 0 is a root, of the aux. eq, of multiplicity 2, the minimal form of y_p will be

$$y_p = \underbrace{(a_0 + a_1 x + a_2 x^2 + a_3 x^3)}_{\text{polynomial of degree } \underline{3}} \cdot \underset{\substack{\uparrow \\ x^m \\ \underline{m=2}}}{x^2} \cdot \underset{\substack{\uparrow \\ e^{\alpha x}, \; \alpha = 0}}{e^{0x}}$$

(b) We have $(D-1)^2(D+1)^2(D-i)(D+i)y = 0$. So

$y_c = (C_1+C_2x)e^{x} + (C_3+C_4x)e^{-x} + C_5\cos x + C_6 \sin x$.

Notice again that the order of the ODE is 6, so we got 6 lin. indep. solutions & 6 arbitrary constants

Now on the RHS we have two terms

$$\underbrace{6x^2}_{\text{polynomial of deg 2}} e^{0x} \leftarrow \begin{matrix}\alpha=0\\ m=0\end{matrix} \quad \text{and} \quad \underbrace{1}_{\text{polynomial of deg. 0}} e^{1.x} \leftarrow \begin{matrix}\alpha=1\\ m=2\end{matrix}$$

So the minimal form of a particular solution will be

$$y_c = \underbrace{(a_0 + a_1 x + a_2 x^2)}_{\text{polynomial of deg 2}} \cdot \underset{m=0}{x^0} \cdot \underset{\alpha=0}{e^{0x}} + \underbrace{(b_0)}_{\text{polynomial of degree 0}} \cdot \underset{m=2}{x^2} \cdot \underset{\alpha=1}{e^{1.x}}$$

(c) Homog. ODE is $(D+3)^2 \cdot (D-2)^3 y = 0$. So

$y_c = (C_1 + C_2x + C_3x^2)e^{2x} + (C_4 + C_5x) \cdot e^{-3x}$.

Order of ODE is 5, so we should have 5 lin. indep. solutions.

Now on the RHS we have $\underbrace{5x}_{\text{polynomial of deg. 1}} \cdot e^{2x} \leftarrow \alpha=2,\ m=3$

So the minimal form of a particular solution will be

$$y_c = \underbrace{(a_0 + a_1 x)}_{\text{polynomial of deg 1}} \cdot \underset{m=3}{x^3} \cdot \underset{\alpha=2}{e^{2x}}.$$

(d) Homog. ODE is $(D-i)^2(D+i)^2(D+1)^3 y = 0$, So

$y_c = (C_1 + C_2x + C_3x^2) \cdot e^{-x} + (C_4 + C_5x)\cos x + (C_6 + C_7 x)\sin x$

RHS is $\underbrace{6x}_{\text{polynomial of deg 1}} \cdot \underbrace{\sin x}_{\alpha = \pm i,\ m=2}$, so we need to add $\cos x$ to the mix

Minimal form of $y_p = (a_0 + a_1x) \cdot x^2 \cdot \cos x + (b_0 + b_1 x) \cdot x^2 \cdot \sin x$

(e) Homog. ODE is $[D-(1+2i)]^2 [D-(1-2i)]^2 \cdot (D-i)(D+i) y = 0$

$\therefore y_c = C_1 \cos x + C_2 \sin x + (C_3 + C_4x)e^x \cos(2x) + (C_5 + C_6x)e^x \sin(2x)$

RHS is $x \cdot e^x \cos(2x)$, so we need to add $e^x \sin(2x)$ to the mix

$$\text{Minimal form of } y_p = \underbrace{(a_0 + a_1 x)}_{\text{polynomial of deg. 1}} \cdot \underset{m=2}{x^2} \cdot \underbrace{e^x \cos(2x)}_{\alpha = 1 \pm 2i} + (b_0 + b_1 x) \cdot x^2 e^x \sin(2x)$$

§4. The Method of variation of parameters & Reduction of order.

The Method of undetermined coefficients is quite a straight forward way of finding a particular solution of a non-homogeneous linear constant-coefficient ODE — but, unfortunately, it only works when the RHS is of the form $q(x).e^{\alpha x}$ where $q(x)$ is a polynomial in x and α is a real or complex number. It does not apply even to the reasonably simple ODE $y''+y=\tan x$. For this we need the Method of variation of parameters. Let us give the theorems first and briefly explain how it was obtained. We shall stick to 2nd order linear ODEs

Theorem 6A (The Method of variation of parameters theorem)
Suppose y_1 and y_2 are linearly independent solutions of the linear ODE

$$y'' + a_1(x).y' + a_0(x).y = 0 \quad \cdots (*)$$

Then a particular solution of the non-homog. linear ODE

$$y'' + a_1(x).y' + a_0(x).y = b(x) \quad \cdots (*)$$

can be obtained by putting $y_p = v_1.y_1 + v_2.y_2$ and solving the equations

$$\begin{cases} y_1.v_1' + y_2.v_2' = 0 \\ y_1'.v_1' + y_2'.v_2' = b(x) \end{cases}$$

for v_1' & v_2' and then integrating (and ignoring the constants of integration to get v_1 & v_2 and thereby get one y_p.

Theorem 6B (Cramer's rule for solving for v_1' & v_2' in Thm 6A)
In Theorem 6A we can find v_1' & v_2' by using Cramer's formula

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ b(x) & y_2' \end{vmatrix}}{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_2' & y_2' \end{vmatrix}}_{= \text{Wronskian of } (y_1,y_2) \neq 0}} \quad \text{and} \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & b(x) \end{vmatrix}}{\underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{W(y_1,y_2)}}$$

Remark We call the Method, variation of parameters because $y_c = C_1 \cdot y_1 + C_2 \cdot y_2$ is the general solution of the homog. equation and by making these arbitrary constants (parameters) into functions (i.e., making them vary) we can obtain a particular solution by a clever trick – that is now used often.

Proof (of Theorem 6A): Suppose y_1 & y_2 are lin. indep. solutions of $(*)$: $y'' + a_1(x) \cdot y' + a_0(x) \cdot y = 0$. Let $y_p = v_1 y_1 + v_2 y_2$.

Then $y_p' = (v_1 \cdot y_1' + v_2 \cdot y_2') + (v_1' \cdot y_1 + v_2' \cdot y_2)$.

Now the only condition we impose on v_1 & v_2 was that $v_1 y_1 + v_2 y_2$ should be a solution of $(**)$. Since v_1 & v_2 are two functions, we can try to impose a second condition and see what happens. Let us choose this second condition to be

$$y_1 \cdot v_1' + y_2 \cdot v_2' = 0 \quad \text{----} \quad (1)$$

Then this makes $y_p' = v_1 \, y' + v_2 \cdot y_2' + 0$.

So $y_p'' = v_1 \cdot y_1'' + v_2 \cdot y_2'' + v_1' y_1' + v_2' y_2'$. So $(**)$ becomes

$$\left(v_1 \cdot y_1'' + v_2 y_2'' + v_1' \cdot y_1' + v_2' \cdot y_2'\right)$$
$$+ a_1(x) \cdot [v_1 \cdot y_1' + v_2 \cdot y_2'] + a_0(x) \cdot [v_1 y_1 + v_2 y_2] = b(x)$$

$$\therefore \; v_1 \underbrace{\left[y_1'' + a_1(x) \cdot y_1' + a_0(x)\, y_1 \right]}_{= 0 \text{ bec. } y_1 \text{ is a sol. of } (*)}$$
$$+ \; v_2 \underbrace{\left[y_2'' + a_1(x) \cdot y_2' + a_0(x)\, y_2 \right]}_{= 0 \text{ bec. } y_2 \text{ is a sol. of } (*)} + \left(v_1' \cdot y_1' + v_2' \cdot y_2'\right) = b(x)$$

$$\therefore \; v_1' \cdot y_1' + v_2' \cdot y_2' = b(x). \quad \text{----} \quad (2)$$

So if we can find v_1 & v_2 such that equations (1) & (2) are satisfied, then we will most likely get a solution $y_p = v_1 y_1 + v_2 y_2$ of the non-homog. equation $(**)$. As always we need to check that we indeed have a solution – and a more complete proof will assure us that under certain conditions, we will always get a particular solution.

Ex. 1. Find a particular solution of the ODE

$$y'' + y = \underbrace{\sec^2 x}_{b(x)}. \qquad (**)$$

Sol. The corresponding homog. ODE is $y'' + y = 0$ $(*)$.

$\therefore (D^2+1)y = 0$. So $y_1 = \cos x$ & $y_2 = \sin x$ are two linearly independent solutions of the homog. ODE. Now from Theorem 6B if we put $y_p = v_1 . y_1 + v_2 y_2$, we get

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ b(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0 - \sin x . \sec^2 x}{\cos^2 x + \sin^2 x}$$

$$= \underbrace{-\frac{\sin x}{\cos x} . \sec x}_{1}$$

$$= -\tan x . \sec x .$$

$$\therefore v_1 = \int -\sec x \tan x \, dx = -\sec x + C_1$$

Also

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & b(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x . \sec^2 x - 0}{\cos^2 x + \sin^2 x}$$

$$= \underbrace{\sec x}_{1}$$

$$\therefore v_2 = \int \sec x \, dx = \ln(\sec x + \tan x) + C_2 .$$

$$\therefore y_p = v_1 y_1 + v_2 y_2 = (-\sec x + C_1)\cos x + [\ln(\sec x + \tan x) + C_2] . \sin x$$

$$= C_1 \cos x + C_2 \sin x + \sin x \ln(\sec x + \tan x) - 1$$

Notice the constants of integration only reproduces the solution y_c of the homog. ODE — so we can always ignore it. If we set $C_1 = C_2 = 0$, then we will get the particular solution $y_p = \sin x . \ln(\sec x + \tan x) - 1$.

Let us check. $y_p' = \cos x . \ln(\sec x + \tan x) + \overbrace{\sin x . \sec x}^{=\tan x} - 0$

$$y_p'' = -\sin x . \ln(\sec x + \tan x) + \underbrace{\cos x . \sec x}_{} + \sec^2 x$$

$$\therefore y_p'' + y_p = -\sin x . \ln(\sec x + \tan x) + 1 + \sec^2 x$$

$$+ \sin x . \ln(\sec x + \tan x) - 1 = \sec^2 x . \checkmark$$

Ex.2 Find a particular solution of the ODE

$$y'' - 2y' + y = e^x/x^2. \qquad (**)$$

by using the Method of variation of parameters

Sol. The homog. ODE is $y'' - 2y' + y = 0 \cdots (*)$. So $(D^2 - 2D + 1)y = 0$. $\therefore (D-1)^2 y = 0$. $\therefore y_c = C_1 e^x + C_2 x e^x$

Let us therefore take $y_1 = e^x$ and $y_2 = xe^x$ as our two linearly indep. sol. of $(*)$. Then if we put $y_p = v_1 y_1 + v_2 y_2$

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ b(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & xe^x \\ e^x/x^2 & (x+1)e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}} = \frac{0 - e^{2x}/x}{(x+1)e^{2x} - xe^{2x}} = \frac{-1}{x}$$

$\therefore \quad v_1 = \int -\frac{1}{x}\,dx = -\ln(x)$. (ignore the const. of integration)

Also
$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & b(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x/x^2 \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}} = \frac{(e^{2x}/x^2) - 0}{(x+1)e^{2x} - xe^{2x}} = \frac{1}{x^2}$$

$\therefore \quad v_2 = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$. (ignore the constant of integration)

$$\therefore \quad y_p = v_1 y_1 + v_2 y_2 = (-\ln x).e^x + \left(\frac{-1}{x}\right).(x.e^x)$$
$$= -(1 + \ln x)\,e^x.$$

Check
$$y_p' = -(1 + \ln x).e^x - \left(0 + \frac{1}{x}\right)e^x$$
$$y_p'' = -(1 + \ln x).e^x - \frac{1}{x}.e^x - \frac{1}{x}.e^x + \frac{1}{x^2}e^x$$

$$\therefore \quad y_p'' - 2y_p' + y_p = -(1+\ln x)e^x - \frac{2}{x}e^x + \frac{1}{x^2}e^x$$
$$+ 2(1+\ln x)e^x + \frac{2}{x}e^x - (1+\ln x).e^x$$
$$= e^x/x^2. \checkmark$$

Theorem 7A (Reduction of order of homog. linear ODE Theorem)

Suppose $y = f(x)$ is a non-trivial solution of the homog. linear ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0 \quad \dots (*)$$

Then the transformation $y = v.f(x)$ reduces $(*)$ into an $(n-1)$th order homog. linear ODE in the dependent variable $w = v'$.

Ex. 3 Consider the ODE $y''' + 2x.y' - 4y = 0 \; \dots (*)$

Check that $y = x^2$ is a solution of $(*)$ and use this solution to reduce the order of $(*)$ by 1.

Sol. Let $f(x) = x^2$. Then $f'(x) = 2x$, $f''(x) = 2$ & $f'''(x) = 0$.

So $f'''(x) + 2x.f'(x) - 4.f(x) = 0 + 2x.(2x) - 4.x^2 \equiv 0$

Hence $y = f(x)$ is a non-trivial solution of $(*)$. [Non-trivial means $y \not\equiv 0$.]

Now put $y = v.x^2$. Then $y' = 2x.v + v'.x^2$,

and $y'' = 2.v + 2x.v' + v'.2x + v''.x^2$

$= 2v + 4x.v' + v''.x^2$

So $y''' = 2v' + 4.v' + 4x.v'' + 2x.v'' + v'''.x^2$

$= 6v' + 6x.v'' + x^2 v'''$.

$\therefore$ $(*)$ becomes

$$(x^2 v''' + 6xv'' + 6v') + 2x.(2x.v + x^2.v') - 4x^2.v = 0$$

$\therefore$ $x^2 v''' + 6xv'' + (2x^3 + 6)v' = 0$. And if we put $w = v'$

we get $x^2 w'' + 6x.w' + (2x^3 + 6)w = 0$ which reduces the order of the original ODE by 1.

Unfortunately, we cannot easily solve this second order linear homog. ODE — but if we started out with a 2nd order homog. linear ODE, we will have a sure way of solving the resulting linear ODE by using an integrating factor.

Theorem 7B (Reduction of order Theorem for 2nd order ODEs)

Let $f(x)$ be a non-trivial solution of the homog. linear ODE

$$y'' + a_1(x).y' + a_0(x).y = 0 \quad \cdots \quad (*)$$

Then the general solution of $(*)$ is $y = C_1 f(x) + C_2 . v . f(x)$.

where $v = \int \dfrac{e^{-\int a_1(x)dx}}{[f(x)]^2} dx$.

Proof: Let $y = v.f(x)$. Then $y' = v.f'(x) + v'.f(x)$. So

$y'' = v.f''(x) + 2v'.f'(x) + v''.f(x)$. Hence $(*)$ becomes

$$[v.f''(x) + 2f'(x).v' + f(x).v''] + a_1(x).[f(x).v' + v.f'(x)] + a_0(x)f(x).v = 0$$

$$\therefore \; v\underbrace{[f''(x) + a_1(x).f'(x) + a_0(x)f(x)]}_{=0 \text{ because } y=f(x) \text{ is a solution of } (*)} + v'[2f'(x) + a_1(x)f(x)] + f(x).v'' = 0$$

$\therefore \; f(x)v'' + [2f'(x) + a_1(x)f(x)]v' = 0$. Now put $w = v'$

Then $\quad f(x).w' + [2f'(x) + a_1(x)f(x)]w = 0$

$\therefore \quad w' = -[2f'(x) + a_1(x)f(x)]w/f(x) = 0$

$\therefore \quad \dfrac{dw}{w} = \left[-\dfrac{2f'(x)}{f(x)} - a_1(x)\right]dx$

$\therefore \quad \ln(w) = -2\ln[f(x)] - \int a_1(x)dx + C$

$\therefore \quad e^{\ln(w)} = e^{\ln[f(x)^{-2}]} . e^{-\int a_1(x)dx} . e^C$

$\therefore \quad v' = w = A . \dfrac{1}{[f(x)]^2} . e^{-\int a_1(x)dx}$ where $A = e^C$

$\therefore \quad v = A.\int \dfrac{e^{-\int a_1(x)dx}}{[f(x)]^2} dx$

Since we only need one particular function v,
we can take $C = 0$, which makes $A = 1$.
So $\quad v = 1.\int \dfrac{e^{-\int a_1(x)dx}}{[f(x)]^2} dx$.

Since $f(x)$ and $v.f(x)$ will be two linearly indep. solutions, the general solution of $(*)$ will then be
$y = C_1 . f(x) + C_2 . v(x) . f(x)$. □

Note: v can never be a constant, because $\dfrac{e^{-\int a_1(x)dx}}{[f(x)]^2}$ is non-trivial.

Ex. 4 Consider the ODE $y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0$. $(*)$
Show that $y = x$ is a solution of $(*)$ and use the Reduction Theorem to find the general solution of $(*)$

Sol. Let $f(x) = x$. Then $f'(x) = 1$ and $f''(x) = 0$
So $f''(x) + \frac{2}{x} f'(x) - \frac{2}{x^2} f(x) = 0 + \frac{2}{x}.1 - \frac{2}{x^2}.x = 0$.
Hence $y = f(x)$ is indeed a solution of $(*)$

If we put $v = \int \frac{e^{-\int a_1(x)\,dx}}{[f(x)]^2}\,dx$, then the general solution will be $y = C_1.f(x) + C_2.v.f(x)$.

Now $v = \int \frac{e^{-\int \frac{2}{x}dx}}{[x]^2}\,dx = \int \frac{e^{-2\ln x}}{[x]^2}\,dx$

$= \int \frac{e^{\ln x^{-2}}}{x^2}\,dx = \int \frac{x^{-2}}{x^2}\,dx = \int x^{-4}\,dx$

$= \frac{x^{-3}}{-3} = -\frac{1}{3}x^{-3}$. (ignore the const. of integr.)

$\therefore$ general solution of $(*)$ is given by

$y = C_1.f(x) + C_2.\left(-\frac{1}{3}x^{-3}\right).f(x)$

$= C_1.x + C_2.\frac{x^{-3}.x^1}{-3}$

$= C_1 x + \frac{C_2}{-3}.x^{-2}$

$= A_1 x + A_2.x^{-2}$ where $A_1 = C_1$ & $A_2 = \frac{-C_2}{3}$.

END OF Ch. 3