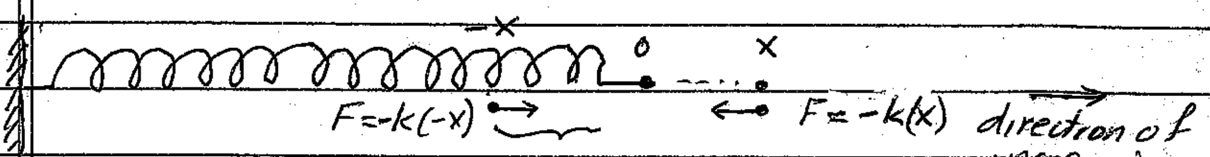
# Ch.4 - Applications of Higher-order Linear ODEs

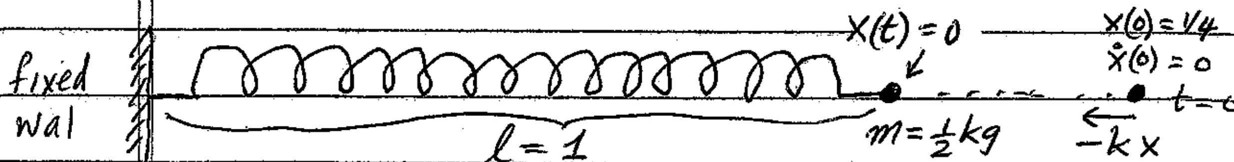
§1. Our main application in this chapter will be concerned with the way a body, which is anchored to spring, moves. These kinds of problems illustrate the different kinds of solutions to the general 2nd order constant-coefficient ODE that exist in nature.

<u>Def</u> A spring is called a <u>Hooke-type</u> spring if it pulls back (or pushes outward) with a force that is directly proportion to the amount it is extended (or contracted) from its natural length.




Most metal springs are approximately Hooke-type.

<u>Ex.1</u> A body of mass (1/2) kg is attached to a Hooke-type spring of natural length 1 m and with spring constant $k = (1/8)\,Nm^{-1}$ on a horizontal frictionless surface. If the other end of the spring is anchored to a fixed walk and the spring is extended from its natural length by (1/4) m at time $t = 0$ sec., find the position of the body at all subsequent times.




<u>Sol.</u> We have from Newton's 2nd law that

$m\ddot{x} = -kx$, where $x(t)$ = amount the spring is extended at time $t$. Here $\ddot{x} = \frac{d^2x}{dt^2}$.

Ex.1 So $(1/2)\ddot{x} + (1/8)x = 0$. $\therefore$ $\ddot{x} + (1/4)x = 0$

$\therefore$ $(D^2 + 1/4)x = 0$. So $D = \pm(i/2)$. Hence

$x(t) = A.\cos(t/2) + B\sin(t/2)$.

$\therefore$ $\dot{x}(t) = -(A/2)\sin(t/2) + (B/2)\cos(t/2)$.

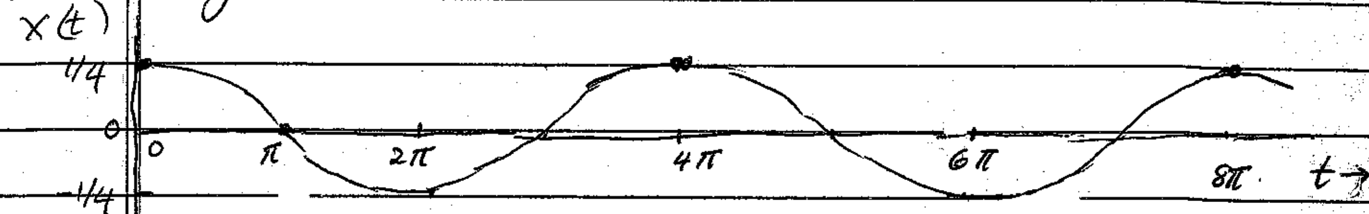
But $x(0) = 1/4$, so $A.\cos(0) + B\sin(0) = 1/4$

$\therefore$ $A + 0 = 1/4 \Rightarrow A = 1/4$.

Also $\dot{x}(0) = 0$, So $-(A/2)\sin(0) + (B/2)\cos(0) = 0$

$\therefore$ $0 + (B/2) = 0 \Rightarrow B = 0$

$\therefore$ $x(t) = (1/4)\cos(t/2)$. Let us also sketch $x(t)$ against $t$ for $t \geq 0$.




This is simple Harmonic motion with amplitude $A = (1/4)$m and period $= 2\pi/(1/2) = 4\pi$ sec.

Ex.2 A body of mass 2kg is attached to a Hooke-type at one end, on a horizontal frictionless surface, and the other end is attached to a fixed vertical wall. The natural length of the spring is 5m, the spring constant is 10 $Nm^{-1}$, and the who apparatus is immersed in a fluid. Also the fluid resistance is $\lambda\dot{x}$ where $v = \dot{x}$ is the velocity of the body and $\lambda = 4\ Nsm^{-1}$. If the spring is extended horizontally by 2m at time $t = 0$ sec, find the position of the particle at all subsequent times.

Sol Let $x(t)$ = the amount the spring is extended at time $t$. Then $m\ddot{x} = -kx - \lambda\dot{x}$ by Newton's 2nd law.

So $2\ddot{x} + 4\dot{x} + 10x = 0$. $\therefore$ $\ddot{x} + 2\dot{x} + 5x = 0$.

Ex. 2

So $(D^2+2D+5)\,x = 0$.

$\therefore\ D = \left[-2 \pm \sqrt{4-4(1)(5)}\right]/2 = -1 \pm 2i$.

$\therefore\ x(t) = A e^{-t}.\cos(2t) + B e^{-t}.\sin(2t)$.

$\therefore\ \dot{x}(t) = -Ae^{-t}.\cos(2t) - 2Ae^{-t}.\sin(2t)$
$\qquad + 2Be^{-t}.\cos(2t) - B.e^{-t}.\sin(2t)$
$\qquad = (2B-A).e^{-t}\cos(2t) - (B+2A).e^{-t}.\sin(2t)$.

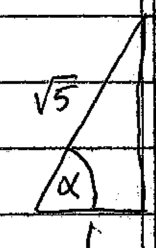
But $x(0)=2$, so $2 = A.e^{-0}\cos(0) + Be^{-0}\sin(0)$
$\therefore\ 2 = A + 0$, so $A = 2$.

Also $\dot{x}(0) = 0$, so $0 = (2B-A)e^{-0}\cos(0) - (B+2A)e^{-0}\sin(0)$
$\therefore\ 0 = (2B-A).1 - 0 \Rightarrow B = A/2 = 1$.

$\therefore\ x(t) = 2e^{-t}\cos(2t) + e^{-t}\sin(2t)$.

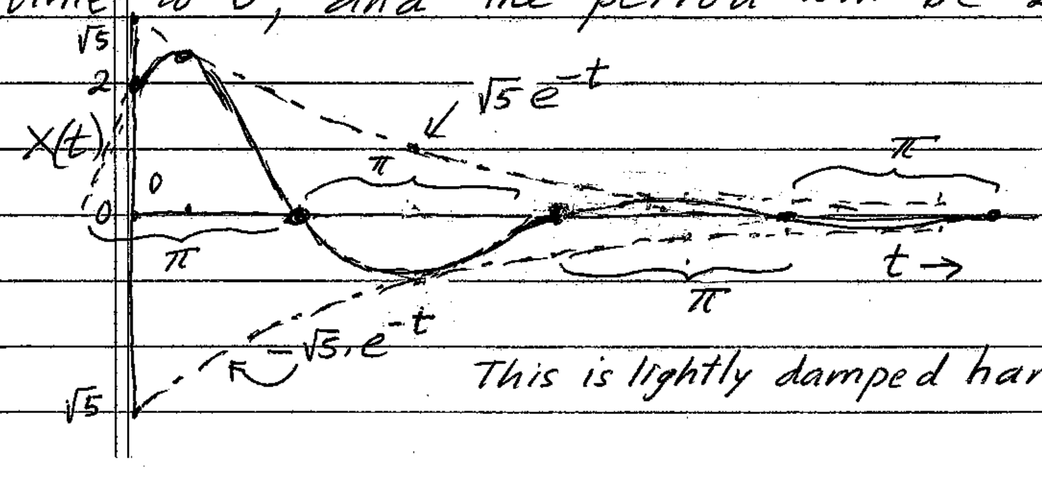
Now let us put $x(t)$ in an amplitude/period format so that we can sketch $x(t)$ against $t$.

$$x(t) = e^{-t}\left[2\cos(2t) + 1.\sin(2t)\right]$$
$$= e^{-t}\sqrt{2^2+1^2}\left[\frac{2}{\sqrt{2^2+1^2}}\cos(2t) + \frac{1}{\sqrt{2^2+1^2}}\sin(2t)\right]$$
$$= e^{-t}.\sqrt{5}.\left[\frac{2}{\sqrt{5}}\cos(2t) + \frac{1}{\sqrt{5}}\sin(2t)\right]$$
$$= \sqrt{5}.e^{-t}\left[\sin\alpha\cos(2t) + \cos(\alpha)\sin(2t)\right]$$
$$= \sqrt{5}.e^{-t}.\sin(\alpha+2t)$$
$$= \sqrt{5}.e^{-t}\sin(2t+\alpha) \quad \text{where } \alpha = \tan^{-1}(2)$$



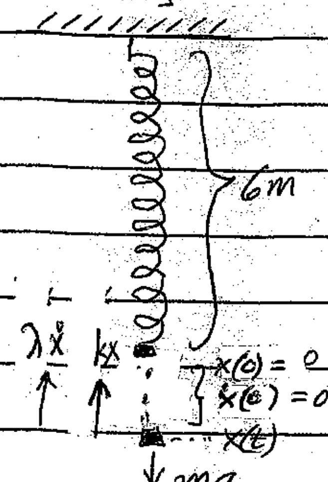

$\tan\alpha = 2$
$\Rightarrow \alpha = \tan^{-1}(2)$.

So the amplitude will be $\sqrt{5}.e^{-t}$, which decreases over time to 0, and the period will be $2\pi/2 = \pi$.




This is lightly damped harmonic motion.

**Ex. 3** A body of mass 3kg is suspended from a very high ceiling by a weightless spring of length 6m. and with spring constant $k = 15\ Nm^{-1}$. The air resistance is $\lambda \dot{x}$ where $\lambda = 12\ Nsm^{-1}$ and the body is let loose at time $t=0$ with the spring un-extended. (a) Find the position of the body at all times. (b) How far below the ceiling will the body eventually come to rest? [Use $g = 10\ ms^{-2}$]




**Sol.** Let $x(t)$ = the amount the spring is extended at time $t = 0$ sec. By Newton's 2nd law $m\ddot{x} = -kx - \lambda\dot{x} + mg$.

So $3\ddot{x} + 12\dot{x} + 15x = 3(10)$

$\therefore \quad \ddot{x} + 4\dot{x} + 5x = 10. \qquad (**)$

The homog. ODE is $(D^2+4D+5)x = 0$

So $D = [-4 \pm \sqrt{16-4(1)(5)}]/2 = -2 \pm i$.

$\therefore\ x_c(t) = Ae^{-2t}\cos(t) + Be^{-2t}\sin(t)$.

Try $x_p(t) = b$ because $10 = 10.e^{0x}$ is a polynomial of deg. 0.

Then $\dot{x}_p(t) = 0$ & $\ddot{x}_p(t) = 0$. So $(**)$ becomes

$0 + 4(0) + 5b = 10 \Rightarrow b = 2$.

So $x_p(t) = 2$. Hence $x(t) = x_c(t) + x_p(t)$

$= Ae^{-2t}\cos(t) + Be^{-2t}\sin(t) + 2$.

$\dot{x}(t) = -2Ae^{-2t}\cos(t) - A.e^{-2t}\sin(t)$
$\quad + Be^{-2t}\cos(t) - 2Be^{-2t}\sin(t) + 0$
$= (B-2A).e^{-2t}\cos(t) - (2B+A)e^{-2t}\sin(t)$

$x(0) = 0 \Rightarrow 0 = A.e^0.\cos(0) + B.e^0.\sin(0) + 2 \Rightarrow A = -2$

$\dot{x}(0) = 0 \Rightarrow 0 = (B-2A).e^0.\cos(0) - (2B+A)e^0.\sin(0) \Rightarrow B = 2A$

(a) $\therefore\ x(t) = 2 - 2e^{-2t}(\cos t + 2\sin t)$ for $t \geq 0$.

(b) $\lim_{t\to\infty} x(t) = 2 - 2.(0) = 2$. So body will settle 8 m below ceiling.

## §2. The physical nature of the ODE $\ddot{x}+2b\dot{x}+\lambda^2 x = F(t)$.

In many applications in Mechanics & Electrical Engineering, we end up with an ODE of the form $\ddot{x}+2b\dot{x}+\lambda^2 x = F(t)$ where $b \geq 0$ and $\lambda > 0$.
If $F(t) \equiv 0$, we say the system is unforced.

Theorem 1 (Unforced Motion Theorem)
If $b \geq 0$ and $\lambda > 0$, then the homogenous linear ODE $\ddot{x}+2b\dot{x}+\lambda^2 x = 0$ has three types of solutions.

(a) If $b^2-\lambda^2 > 0$, the system is said to be over-damped and the solution is of the form

$$x(t) = C_1 e^{-t(b-\sqrt{b^2-\lambda^2})} + C_2 e^{-t(b+\sqrt{b^2-\lambda^2})}.$$

(b) If $b^2-\lambda^2=0$, the system is said to be critically damped and the solution is of the form

$$x(t) = (C_1 + C_2 t)e^{-bt}.$$

(c) If $b^2-\lambda^2<0$, the system is said to be under-damped and the solution is of the form

$$x(t) = e^{-bt}\left[C_1 \cos(t\sqrt{\lambda^2-b^2}) + C_2 \sin(t\sqrt{\lambda^2-b^2})\right]$$

Justification: The auxiliary equation of $\ddot{x}+2b\dot{x}+\lambda^2 x=0$ is just $D^2+2bD+\lambda^2=0$. So in

Case (a): $D = (-2b \pm \sqrt{4b^2-4\lambda^2})/2$
$= -b \pm \sqrt{b^2-\lambda^2}$ & the result follows

Case (b) $D = (-2b \pm \sqrt{4b^2-4b^2})/2 = -b$ (twice) and the result for case (b) follows.

Case (c) $D = (-2b \pm i\sqrt{4\lambda^2-4b^2})/2$
$= -b \pm i\sqrt{\lambda^2-b^2}$ & the result follows.

Theorem 2 (Forced Motion Theorem)

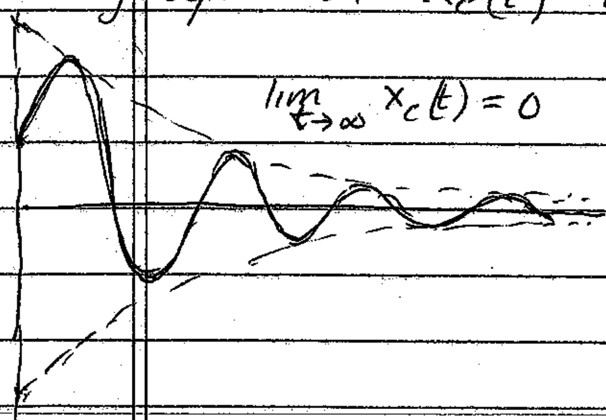
If $b>0$ and $\lambda>0$, the non-homogeneous linear ODE $\ddot{x}+2b\dot{x}+\lambda^2 x = F(t)$ has a general solution of the form $x(t) = x_c(t) + x_p(t)$, where $x_c(t)$ is the solution of the corresponding homogeneous ODE. Since $b>0$, $\lim\limits_{t\to\infty} x_c(t) = 0$.

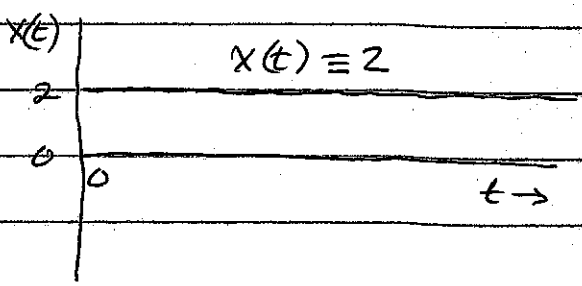
For this reason, $x_c(t)$ is called the transient part of the solution and $x_p(t)$ is called the steady-state part of the solution.

Ex. 1 In example 3 of section 1, we found that

$$x(t) = \underbrace{-2e^{-2t}[\cos(t)+2\sin(t)]}_{x_c(t)\,=\,\text{transient part}} + \underbrace{2}_{x_p(t)\,=\,\text{steady state part}}$$

The graphs of $x_c(t)$ & $x_p(t)$ are as shown below




Theorem 3 (Forced Oscillatory Motion Theorem)

If $b>0$; $\lambda>0$, and $\lambda^2-b^2>0$, then the non-homogeneous ODE $\ddot{x}+2b\dot{x}+\lambda^2 x = E\cdot\cos(\omega t)$ (where $E>0$ and $\omega>0$ are constants) has a solution of the form $x(t) = x_c(t)+x_p(t)$ where

$$x_c(t) = A\cdot e^{-bt}\cdot\cos\left(t\sqrt{\lambda^2-b^2}+\varphi\right), \text{ and}$$

$$x_p(t) = \left[E\Big/\sqrt{(\lambda^2-b^2)^2+4\omega^2 b^2}\right]\cos(\omega t-\theta),$$

and $A$ is called the initial amplitude, and $\varphi$ & $\theta$ are called the phase-shifts.

Now let us examine the function

$$A(\omega) = E/\sqrt{(\lambda^2-\omega^2)^2+4\omega^2b^2}$$
$$= E.\left[(\lambda^2-\omega^2)^2+4\omega^2b^2\right]^{-1/2}.$$

Then

$$A'(\omega) = -(E/2).\left[2(\lambda^2-\omega^2)(-2\omega)+4b^2.2\omega\right].\left[(\lambda^2-\omega^2)^2+4\omega^2b^2\right]^{-3/2}$$
$$= -(E/2).(-4\omega).\left[(\lambda^2-\omega^2)-2b^2\right].\{A(\omega)/E\}^3$$
$$= \underbrace{(2\omega)}_{>0}.\underbrace{\{A(\omega)^3/E^2\}}_{>0}.\left[(\lambda^2-2b^2)-\omega^2\right]$$

Now the only term that can be zero is $\boxed{(\lambda^2-2b^2)-\omega^2}$. Also if $\lambda^2-2b^2 \le 0$, then this term will be negative and so $A(\omega)$ will be monotonically decreasing on $(0,\infty)$ and hence will be very boring.

However if $\lambda^2-2b^2>0$, then $A'(\omega)=0$ when $\omega^2=\lambda^2-2b^2$, i.e., when $\omega=\sqrt{\lambda^2-2b^2}$ and we check that this will be a local maximum.

So put $\omega_1=\sqrt{\lambda^2-2b^2}$ and assume $\lambda^2>2b^2$. Then $\omega_1$ will be a local maximum of $A(\omega)$ and

$$A(\omega_1) = E/\sqrt{[\lambda^2-(\lambda^2-2b^2)^2]^2+4b^2.(\lambda^2-2b^2)}$$
$$= E/\left[2b\sqrt{\lambda^2-b^2}\right]$$

Since $\lambda^2>2b^2$, $\lambda^2-b^2>2b^2-b^2=b^2$

$\therefore$ $A(\omega_1) > E/2b(b) = E/(2b^2)$.

Hence if $b$ is very small, then $A(\omega_1)$ will be very, very large.

If $\omega=\omega_1$, the forcing function $F(t) = E.\cos(\omega t)$ is said to be in <u>resonance</u> with the system. When this happens the maximum amplitude of all possible steady-solutions is attained. When $b$ is fairly small this can result in disaster as with the Tacoma Narrows suspension bridge. But this same resonance can help us tune our systems and find the correct radio stations by controlling $b$.