Ch.5 - The Laplace transform & its applications (1)

In this chapter we will introduce a linear operator called the Laplace transform, study its properties, and use it to solve linear constant-coefficient ODEs. The Laplace transform takes a function $f(t)$ [t = time] transforms into a function $F(s)$ [s = frequency].

§1. Finding Laplace transforms of various kinds of function

<u>Def.</u> A function $f(t)$ with domain $[0,\infty)$ is of <u>exponential order</u> if we can find constants α, $M \geq 0$ such that for all $t \in [0,\infty)$, $|f(t)| \leq M \cdot e^{\alpha t}$.

<u>Def.</u> Let $f(t)$ be a real-valued function, with domain $(0,\infty)$, which is of exponential order. We define the Laplace transform of $f(t)$ to be the function $F(s)$ defined by

$$F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt,$$

and we write $\mathcal{L}\{f\}(s) = F(s)$.

<u>Ex.1</u> Find the Laplace transform of the function $f:[0,\infty) \to \mathbb{R}$ defined by $f(t) \equiv 1$.

<u>Sol.</u>
$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \lim_{R\to\infty} \int_0^R e^{-st}\, dt$$
$$= \lim_{R\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^R = \lim_{R\to\infty}\left[\frac{1}{s} - \frac{1}{s}e^{-sR}\right] = \frac{1}{s}.$$

<u>Ex.2</u> Find the Laplace transform of the function $f:[0,\infty) \to \mathbb{R}$, $f(t) = e^{at}$ where a is a real constant

<u>Sol.</u>
$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt = \lim_{R\to\infty} \int_0^R e^{-(s-a)t}\, dt$$
$$= \lim_{R\to\infty}\left[\frac{-1}{s-a} e^{-(s-a)t}\right]_0^R = \lim_{R\to\infty}\left[\frac{1}{s-a} - \frac{1}{s-a} e^{-(s-a)R}\right] = \frac{1}{s-a}.$$

<u>Proposition 1</u> Let $f_1(t), f_2(t)$ & $f(t)$ be functions of exponential order that are defined on $(0,\infty)$ and c be a constant. Then

(a) $\mathcal{L}\{f_1 \pm f_2\}(s) = \mathcal{L}\{f_1\}(s) \pm \mathcal{L}\{f_2\}(s)$

(b) $\mathcal{L}\{c.f\}(s) = c.\ \mathcal{L}\{f\}(s)$

(c) $\mathcal{L}\{e^{at}.f(t)\}(s) = \mathcal{L}\{f\}(s-a)$

From (a) & (b), we can see that $\mathcal{L}$ behaves like the differential operator $\mathcal{D}$.

<u>Proof</u> (a) $\mathcal{L}\{f_1 \pm f_2\}(s) = \int_0^\infty e^{-st}.[f_1(t) \pm f_2(t)]\,dt$

$$= \int_0^\infty e^{-st}.f_1(t)\,dt \pm \int_0^\infty e^{-st}.f_2(t)\,dt$$

$$= \mathcal{L}\{f_1\}(s) \pm \mathcal{L}\{f_2\}(s)$$

(b) $\mathcal{L}\{c.f\}(s) = \int_0^\infty e^{-st}.c.f(t)\,dt$

$$= c.\int_0^\infty e^{-st}.f(t)\,dt = c.\ \mathcal{L}\{f\}(s)$$

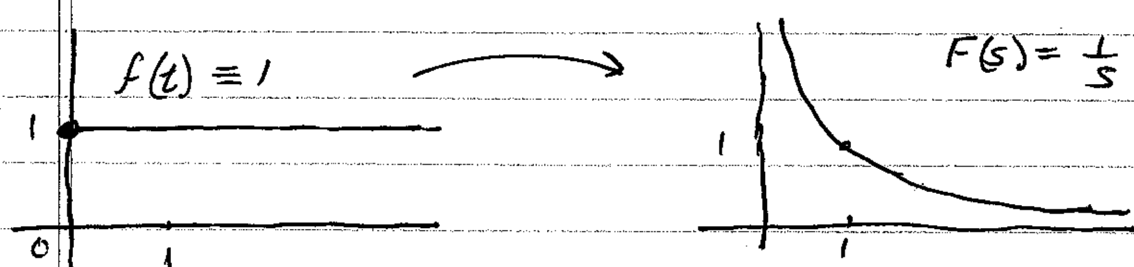
(c) $\mathcal{L}\{e^{at}.f(t)\}(s) = \int_0^\infty e^{-st}.e^{at}.f(t).dt$

$$= \int_0^\infty e^{-(s-a)t}.f(t).dt$$

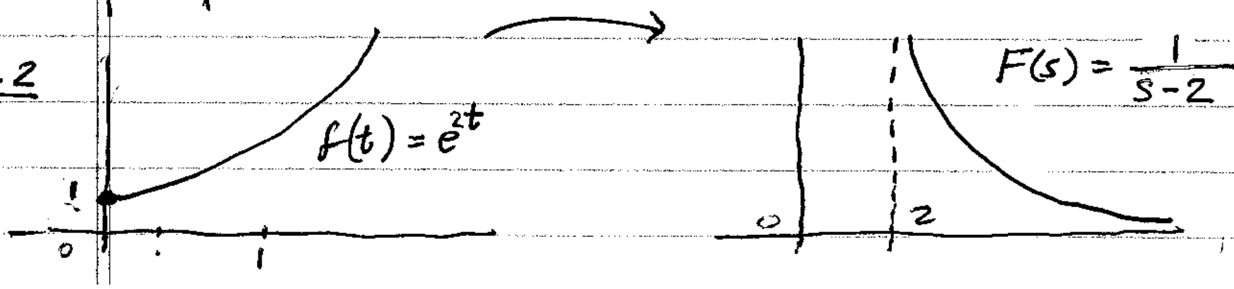
$$= \int_0^\infty e^{-\omega t}.f(t).dt \quad \text{where } \omega = s-a$$

$$= \mathcal{L}\{f\}(\omega) = \mathcal{L}\{f\}(s-a).$$

<u>Ex.1</u>



<u>Ex.2</u>



<u>Proposition 2</u> Let $f(t)$ be a function of exponential order that is defined on $(0,\infty)$ and $F(s) = \mathcal{L}\{f\}(s)$. Then

(a) $\mathcal{L}\{t.f(t)\}(s) = -\frac{d}{ds}[F(s)]$, and in general

(b) $\mathcal{L}\{t^n.f(t)\}(s) = (-1)^n.\frac{d^n}{ds^n}[F(s)]$.

<u>Proof:</u> (a) We have $F(s) = \int_0^\infty e^{-st}.f(t).dt$. So

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st}.f(t)\,dt\right] = \int_0^\infty \frac{d}{ds}\left[e^{-st}.f(t)\,dt\right]$$

by a theorem on differentiating under an integral sign

$$= \int_0^\infty (-t.e^{-st}).f(t).dt$$

$$= -\int_0^\infty e^{-st}.[t.f(t)].dt$$

$$= -\mathcal{L}\{t.f(t)\}(s).$$

$$\therefore\ \mathcal{L}\{t.f(t)\}(s) = -\frac{d}{ds}[F(s)] = -F'(s).$$

(b) $$\frac{d^n}{ds^n}[F(s)] = \left(\frac{d}{ds}\right)^n \int_0^\infty e^{-st}.f(t)\,dt$$

$$= \int_0^\infty \frac{d^n}{ds^n}\left[e^{-st}.f(t)\right].dt = \int_0^\infty \left[(-t)^n.e^{-st}.f(t)\right]dt$$

$$= (-1)^n.\int_0^\infty e^{-st}.t^n.f(t).dt = (-1)^n.\mathcal{L}\{t^n.f(t)\}(s).$$

$$\therefore\ \mathcal{L}\{t^n.f(t)\}(s) = \frac{1}{(-1)^n}\frac{d^n}{ds^n}F(s) = \frac{(-1)^{2n}}{(-1)^n}.F^{(n)}(s) = (-1)^n F^{(n)}(s).$$

<u>Ex. 3</u> Find the Laplace transform of the following functions

(a) $f(t) = t$ (b) $f(t) = t^2$ (c) $f(t) = t^n$.

<u>Sol</u> (a) $\mathcal{L}\{t\}(s) = \mathcal{L}\{t.1\}(s) = -\frac{d}{ds}[\mathcal{L}\{1\}(s)]$

$$= -\frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{d}{ds}(s^{-1}) = -(-1)s^{-2} = \frac{1}{s^2}$$

(b) $\mathcal{L}(t^2) = \mathcal{L}\{t.t\}(s) = -\frac{d}{ds}[\mathcal{L}\{t\}(s)] = -\frac{d}{ds}\left(\frac{1}{s^2}\right)$

$$= -\frac{d}{ds}(s^{-2}) = -(-2).s^{-3} = 2/s^3.$$

(c) $\mathcal{L}\{t^n\}(s) = \mathcal{L}\{t^n \cdot 1\}(s) = (-1)^n \dfrac{d^n}{ds^n}\left[\mathcal{L}\{1\}(s)\right]$

$= (-1)^n \cdot \dfrac{d^n}{ds^n}(s^{-1}) = (-1)^n \cdot (-1)^n \cdot n! \; s^{-(n+1)} = \dfrac{n!}{s^{n+1}}.$

Ex. 4 Find the Laplace transform of the following functions.

(a) $t e^{at}$ (b) $t^2 e^{at}$

Sol (a) $\mathcal{L}\{t \cdot e^{at}\}(s) = \mathcal{L}\{e^{at} \cdot t\}(s) = \{\mathcal{L}(t)\}(s-a)$

$= \dfrac{1}{(s-a)^2}$, bec. $\mathcal{L}\{t\}(s) = \dfrac{1}{s^2}$

(b) $\mathcal{L}\{t^2 \cdot e^{at}\}(s) = \mathcal{L}\{e^{at} \cdot t^2\}(s) = \mathcal{L}(t^2)(s-a)$

$= \dfrac{2}{(s-a)^3}$ bec. $\mathcal{L}\{t^2\}(s) = \dfrac{2}{s^3}$.

Ex. 5 Find the Laplace transform of the following functions.

(a) $\cos(at)$ (b) $\sin(bt)$ (c) $t \cdot \sin(bt)$.

Sol (a) $\mathcal{L}\{\cos(at)\}(s) = \mathcal{L}\left\{\dfrac{e^{iat} + e^{-iat}}{2}\right\}(s)$

$= \dfrac{1}{2}\left[\mathcal{L}\{e^{iat}\}(s) + \mathcal{L}\{e^{-iat}\}(s)\right] = \dfrac{1}{2}\left[\dfrac{1}{(s-ia)} + \dfrac{1}{(s--ia)}\right]$

$= \dfrac{1}{2}\left(\dfrac{1}{s-ia} + \dfrac{1}{s+ia}\right) = \dfrac{1}{2} \dfrac{2s}{(s-ia)(s+ia)} = \dfrac{s}{s^2+a^2}$

(b) $\mathcal{L}\{\sin(bt)\}(s) = \mathcal{L}\left\{\dfrac{e^{ibt} - e^{-ibt}}{2i}\right\}(s)$

$= \dfrac{1}{2i}\left[\mathcal{L}\{e^{ibt}\}(s) - \mathcal{L}\{e^{-ibt}\}(s)\right] = \dfrac{1}{2i}\left(\dfrac{1}{s-ib} - \dfrac{1}{s+ib}\right)$

$= \dfrac{1}{2i} \cdot \dfrac{2ib}{(s-ib)(s+ib)} = \dfrac{b}{s^2+b^2}.$

(c) $\mathcal{L}[t \cdot \sin(bt)](s) = -\dfrac{d}{ds}\left[\mathcal{L}\{\sin(bt)\}(s)\right] = -\dfrac{d}{ds}\left[b \cdot (s^2+b^2)^{-1}\right]$

$= (-1) \cdot (-1) \cdot b \cdot 2s \cdot (s^2+b^2)^{-2} = 2bs/(s^2+b^2).$

By using Propositions 1 & 2 we can obtain a table of the Laplace transforms of many functions. You will notice the absence of t^{-n} and $\ln(t)$ — but that is because these are not functions of exponential order.

	$f(t)$	$\mathcal{L}\{f(t)\}(s)$	conditions
0.	0	0	
1.	1	$1/s$	
2.	e^{at}	$1/(s-a)$	$a \in \mathbb{R}$
3.	t^n	$n!/s^{n+1}$	$n \geq 0$
4.	$t^n e^{at}$	$n!/(s-a)^{n+1}$	$a \in \mathbb{R},\ n \geq 0$
5.	$\sin(bt)$	$b/(s^2+b^2)$	$b>0$
6.	$\cos(bt)$	$s/(s^2+b^2)$	$b>0$
7.	$e^{at}.\sin(bt)$	$b/[(s-a)^2+b^2]$	
8.	$e^{at}.\cos(bt)$	$(s-a)/[(s-a)^2+b^2]$	
9.	$t.\sin(bt)$	$2bs/(s^2+b^2)^2$	
10.	$t.\cos(bt)$	$(s^2-a^2)/(s^2+b^2)^2$	
11.	$u_a(t) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t \geq a \end{cases}$	$\dfrac{e^{-as}}{s}$	$a \geq 0$
G1.	$f_1(t) \pm f_2(t)$	$\mathcal{L}\{f_1\}(s) \pm \mathcal{L}\{f_2\}(s)$	
G2.	$c.f_1(t)$, (c const.)	$c.\mathcal{L}\{f_1\}(s)$	
G3.	$e^{at}.f_1(t)$	$\mathcal{L}\{f_1\}(s-a)$	
G4.	$t.f_1(t)$	$-\dfrac{d}{ds}[\mathcal{L}\{f_1\}(s)]$	
G5.	$t^n.f_1(t)$	$(-1)^n.\dfrac{d^n}{ds^n}[\mathcal{L}\{f_1\}(s)]$	
G6.	$f_1'(t) = \frac{d}{dt}[f_1(t)]$	$s.\mathcal{L}\{f_1\}(s) - f_1(0)$	
G7.	$f_1''(t) = \dfrac{d^2}{dt^2}[f_1(t)]$	$s^2.\mathcal{L}\{f_1\}(s) - s.f_1(0) - f_1'(0)$.	
G8.	$(f_1 * f_2)(t)$	$\mathcal{L}\{f_1(t)\}(s).\mathcal{L}\{f_2(t)\}(s)$.	

§2. Using Laplace transform to solve linear ODEs

There is an important theorem that tells us what is the relation between $\mathcal{L}\{f'(t)\}(s)$ & $\mathcal{L}\{f(t)\}(s)$ — and this allows to be able solve certain kinds of linear ODEs.

Theorem 3 Let $f(t)$ be a function of exponential order with domain $[0,\infty)$. Then

(a) $\mathcal{L}\{f'(t)\}(s) = s.[\mathcal{L}\{f(t)\}(s)] - f(0)$.

(b) $\mathcal{L}\{f''(t)\}(s) = s^2.[\mathcal{L}\{f(t)\}(s)] - s.f(0) - f'(0)$.

(c) $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n.[\mathcal{L}\{f(t)\}(s)] - s^{n-1}.f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$.

Proof: (a) $\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st}.f'(t)\,dt$

Put $u = e^{-st}$
Then $du = -s.e^{-st}du$
Put $dv = f'(t)dt$
Then $v = f(t)$

$$= \lim_{R\to\infty}\int_0^R e^{-st}.f'(t)\,dt$$

$$= \lim_{R\to\infty}\int_0^R u\,.\,dv$$

$$= \lim_{R\to\infty}\left\{[u.v]_0^R - \int_0^R v.du\right\} = \lim_{R\to\infty}\left\{[f(t)e^{-st}]_0^R - \int_0^R(-s.e^{-st}.f(t)dt\right\}$$

$$= \lim_{R\to\infty}[f(R).e^{-sR} - f(0).1] + \left[\int_0^\infty s.e^{-st}.f(t)dt\right]$$

$$= s.\int_0^\infty e^{-st}.f(t)dt - f(0) = s[\mathcal{L}\{f(t)\}(s)] - f(0)$$

(b) $\mathcal{L}\{f''(t)\}(s) = \mathcal{L}\{(f')'(t)\}(s)$

$$= s.\ \mathcal{L}[\{f'(t)\}(s)] - f'(0) \quad \text{by (a)}$$

$$= s.\{s.[\mathcal{L}\{f(t)\}(s) - f(0)\} - f'(0)$$

$$= s^2.[\mathcal{L}\{f(t)\}(s)] - s.f(0) - f'(0).$$

(c) This can be done by using Mathematical Induction and the result from part (a) in a similar way that we proved part (b).

Ex. 1 Find the solution of the ODE
$y'(t) - 3.y(t) = 6e^t$ with the initial condition
$y(0) = 2$ by using the Laplace transform.

Sol. $y'(t) - 3.y(t) = 6e^t$

So $\mathcal{L}\{y'(t) - 3.y(t)\} = \mathcal{L}\{6.e^t\}$

$\therefore \mathcal{L}\{y'(t)\} - 3.\mathcal{L}\{y(t)\} = 6.\mathcal{L}\{e^t\}$

$\therefore s.\mathcal{L}\{y\} - y(0) - 3\mathcal{L}\{y\} = 6.\frac{1}{s-1}$

$\therefore (s-3)\mathcal{L}\{y\} = \frac{6}{s-1} + y(0) = \frac{6}{s-1} + 2 = \frac{2s+4}{s-1}$

$\therefore \mathcal{L}\{y\} = \frac{2s+4}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1}$

$\therefore 2s+4 = A(s-1) + B(s-3)$

Putting $s=3$, gives $2(3)+4 = A.(3-1) \Rightarrow A = 5$.
Putting $s=1$, gives $2(1)+4 = B(1-3) \Rightarrow B = -3$.

$\therefore \mathcal{L}\{y\} = \frac{5}{s-3} - \frac{3}{s-1}$.

$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{5}{s-3} - \frac{3}{s-1}\right\} = 5.\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$= 5.e^{3t} - 3.e^t$.

Ex. 2 Find the solution of the ODE
$y''(t) + y'(t) - 2.y(t) = 0$ with $y(0) = 1$ & $y'(0) = 4$.

Sol. We have $\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 2.\mathcal{L}\{y\} = 0$

$\therefore \left[s^2\mathcal{L}\{y\} - s.y(0) - y'(0)\right] + \left[s.\mathcal{L}\{y\} - y(0)\right] - 2.\mathcal{L}\{y\} = 0$

$\therefore (s^2 + s - 2).\mathcal{L}\{y\} = s.y(0) + y'(0) + y(0)$

$\therefore (s-1)(s+2).\mathcal{L}\{y\} = s.1 + 4 + 1 = s+5$

$\therefore \mathcal{L}\{y\} = \frac{s+5}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$

$\therefore s+5 = A(s+2) + B(s-1)$

Putting $s=1$ gives $1+5 = A(s+1) \Rightarrow 6 = 3A \Rightarrow A = 2$

Putting $s=-2$ gives us $-2+5 = B(-2-1) \Rightarrow 3=-3B \Rightarrow B=-1$

$\therefore \mathcal{L}\{y\} = \frac{2}{s-1} - \frac{1}{s+2}$. So $y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{1}{s+2}\right\}$

$= 2 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = 2 \cdot e^{t} - e^{-2t}$.

<u>Ex.3</u> Find the solution of the ODE

$y''(t) - 2y'(t) + 5 \cdot y(t) = 0$ with $y(0) = 3$ & $y'(0) = -1$.

<u>Sol.</u> $\mathcal{L}\{y''\} - 2 \cdot \mathcal{L}\{y'\} + 5\,\mathcal{L}\{y\} = 0$

$\therefore \left[s^2 \cdot \mathcal{L}\{y\} - s \cdot y(0) - y'(0)\right] - 2\left[s \cdot \mathcal{L}\{y\} - y(0)\right] + 5 \cdot \mathcal{L}\{y\} = 0$

$\therefore (s^2 - 2s + 5)\,\mathcal{L}\{y\} = s \cdot y(0) + y'(0) - 2 \cdot y(0) = 3s - 1 - 6 = 3s - 7$

$\therefore \mathcal{L}\{y\} = (3s-7)/(s^2 - 2s + 5) = (3s-7)/\left[(s-1)^2 + 4\right]$

$\therefore \mathcal{L}\{y\} = \frac{3(s-1) - 4}{(s-1)^2 + 2^2} = 3 \cdot \frac{s-1}{(s-1)^2 + 2^2} - 2 \cdot \frac{2}{(s-1)^2 + 2^2}$

$\therefore y(t) = 3 \cdot \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 2^2}\right\} - 2 \cdot \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2 + 2^2}\right\}$

$= 3 \cdot e^{t} \cdot \cos(2t) - 2 \cdot e^{t} \cdot \sin(2t)$

<u>Ex. 4</u> Find the solution of the ODE

$y''(0) - 6 \cdot y'(t) + 9\,y(t) = 0$ with $y(0) = 4$ & $y'(0) = 3$

<u>Sol.</u> $\mathcal{L}\{y''\} - 6 \cdot \mathcal{L}\{y'\} + 9 \cdot \mathcal{L}\{y\} = 0$

$\therefore \left[s^2 \cdot \mathcal{L}\{y\} - s \cdot y(0) - y'(0)\right] - 6 \cdot \left[s\,\mathcal{L}\{y\} - y(0)\right] + 9 \cdot \mathcal{L}\{y\} = 0$

$\therefore (s^2 - 6s + 9) \cdot \mathcal{L}\{y\} = s \cdot y(0) + y'(0) - 6\,y(0) = 4s - 21$

$\therefore (s-3)^2 \cdot \mathcal{L}\{y\} = 4(s-3) - 9$.

$\therefore \mathcal{L}\{y\} = \frac{4(s-3) - 10}{(s-3)^2} = \frac{4}{s-3} - \frac{9}{(s-3)^2}$

$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{4}{s-3}\right\} - \mathcal{L}\left\{\frac{9}{(s-3)^2}\right\}$

$= 4 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - 9\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$

$= 4 \cdot e^{3t} - 9 \cdot (t \cdot e^{3t}) = (4 - 9t) \cdot e^{3t}$.

§3. Laplace transform solutions of Linear systems of ODEs

In this section we will use the Laplace transform to solve simple systems of linear ODE in more than one dependent variables and in one independent variable t.

Ex.1 Use the Laplace transform to find the solution of the system

$$\begin{cases} x'(t) + y(t) = 2e^t & \text{with } x(0) = 0 \\ y'(t) - x(t) = 0 & \text{and } y(0) = 1. \end{cases}$$

Sol. We have

$$\mathcal{L}\{x'\} + \mathcal{L}\{y\} = 2 . \mathcal{L}\{e^t\}$$
$$\mathcal{L}\{y'\} - \mathcal{L}\{x\} = 0$$

$$\therefore \quad [s.\mathcal{L}\{x\} - x(0)] + \mathcal{L}\{y\} = 2/(s-1)$$
$$[s.\mathcal{L}\{y\} - y(0)] - \mathcal{L}\{x\} = 0$$

$$\therefore \quad s.\mathcal{L}\{x\} + \mathcal{L}\{y\} = 2/(s-1) + x(0) = \frac{2}{s-1} \qquad (1)$$
$$s.\mathcal{L}\{y\} - \mathcal{L}\{x\} = y(0) = 1. \qquad (2)$$

Now from (2) $\mathcal{L}\{x\} = s.\mathcal{L}\{y\} - 1$. Subst. in (1) gives us

$$s.[s.\mathcal{L}\{y\} - 1] + \mathcal{L}\{y\} = \frac{2}{s-1}$$

$$\therefore \quad s^2.\mathcal{L}\{y\} + \mathcal{L}\{y\} = \frac{2}{s-1} + s = \frac{s^2-s+2}{s-1}$$

$$\therefore \quad (s^2+1).\mathcal{L}\{y\} = \frac{s^2-s+2}{s-1} \Rightarrow \mathcal{L}\{y\} = \frac{s^2-s+2}{(s-1)(s^2+1)}$$

$$\therefore \quad \mathcal{L}\{y\} = \frac{s^2-s+2}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$\therefore \quad s^2-s+2 = A(s^2+1) + (Bs+C)(s-1).$$

Putting $s=1$ gives us $1^2-1+2 = A(1+1) \Rightarrow A = 1$.

Putting $s=0$ gives us $0^2-0+2 = A(0+1) + C(0-1)$

$\therefore \quad 2 = A - C \Rightarrow C = A-2 = 1-2 = -1.$

Putting $s=-1$ gives us $1+1+2 = A(1+1) + (C-B)(-1-1)$

$\therefore\ 4 = 1(2) + (-1-B)(-2) \Rightarrow 4 = 4 + 2B \Rightarrow B = 0$

$\therefore\ \mathcal{L}\{y\} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1} = \dfrac{1}{s-1} - \dfrac{1}{s^2+1}$

$\therefore\ y(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = e^t - \sin(t).$

Also from the original equations, we know $y'(t) - x(t) = 0$. So $x(t) = y'(t) = e^t - \cos(t)$.

Ex.2 Use the Laplace transform to find the solution of the system
$$\begin{cases} x'(t) - y(t) = 0 & \text{with } x(0) = 0, \\ y'(t) - x(t) = 2 & \text{and } y(0) = 4. \end{cases}$$

<u>Sol.</u> We have
$$\mathcal{L}\{x'\} - \mathcal{L}\{y\} = 0,$$
$$\mathcal{L}\{y'\} - \mathcal{L}\{x\} = \mathcal{L}\{2\}.$$

$$\therefore\ [s.\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = 0.$$
$$[s.\mathcal{L}\{y\} - y(0)] - \mathcal{L}\{x\} = 2/s.$$

$$\therefore\ s.\mathcal{L}\{x\} - \mathcal{L}\{y\} = x(0) = 0 \quad (1)$$
$$s.\mathcal{L}\{y\} - \mathcal{L}\{x\} = \frac{2}{s} + y(0) = \frac{2}{s} + 4 = \frac{4s+2}{s} \quad (2)$$

From (1), we get $\mathcal{L}\{y\} = s.\mathcal{L}\{x\}$. Subst. in (2) we get
$$s.[s.\mathcal{L}\{x\}] - \mathcal{L}\{x\} = \frac{4s+2}{s}$$
$$\therefore\ (s^2-1).\mathcal{L}\{x\} = \frac{4s+2}{s}$$
$$\therefore\ \mathcal{L}\{x\} = \frac{4s+2}{s(s^2-1)} = \frac{4s+2}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$
$$\therefore\ 4s+2 = A(s-1)(s+1) + Bs(s+1) + C.s(s-1)$$

Putting $s = 0$, gives $4(0) + 2 = A(-1)(1) \Rightarrow A = -2.$

Putting $s = 1$ gives $4 + 2 = 0 + B(1)(2) + 0 \Rightarrow B = 3$

Putting $s = -1$ gives $-4 + 2 = 0 + 0 + C(-1)(-2) \Rightarrow C = -1.$

$\therefore \quad \mathcal{L}\{x\} = \frac{-2}{s} + \frac{3}{s-1} - \frac{1}{s+1}$ (11)

So $\quad x(t) = -2\,\mathcal{L}^{-1}\{\frac{1}{s}\} + 3.\mathcal{L}^{-1}\{\frac{1}{s-1}\} - \mathcal{L}^{-1}\{\frac{1}{s+1}\}$

$\qquad = -2 + 3e^{t} - e^{-t}.$

Also from the original equations, $x'(t) - y(t) = 0$

So we get $y(t) = x'(t) = 3e^{t} + e^{-t}$

<u>Quick check:</u> $x(0) = -2 + 3e^{0} - e^{-0} = 0$ ✓

$\qquad y(0) = 3e^{0} + e^{-0} = 4$ ✓

<u>Ex.3</u> Solve the system $\begin{cases} x'(t) - 2y(t) = 2 \\ y'(t) - 2x(t) = 3e^{-t} \end{cases}$

with $x(0) = -1$ & $y(0) = 1$

<u>Sol.</u> We have $\mathcal{L}\{x'\} - 2\,\mathcal{L}\{y\} = 2/s$

$\qquad \mathcal{L}\{y'\} - 2\,\mathcal{L}\{x\} = 3/(s+1)$

So $\quad s.\mathcal{L}\{x\} - x(0) - 2.\mathcal{L}\{y\} = 2/s$ (1)

and $\quad s.\mathcal{L}\{y\} - y(0) - 2.\mathcal{L}\{x\} = 3/(s+1)$ (2)

$\therefore \quad \mathcal{L}\{y\} = [s.\mathcal{L}\{x\} + 1 - 2/s]/2$ from (1). Sub. in (2)

gives $\quad s.[s.\mathcal{L}\{x\} + 1 - 2/s]/2 - 1 - 2\mathcal{L}\{x\} = 3/(s+1)$

$\therefore \quad \{s^2 - 4\}.\mathcal{L}\{x\} + s - 2 - 2 = 6/(s+1)$

$\therefore \quad \{s^2 - 4\}.\mathcal{L}\{x\} = \frac{4-s}{1} + \frac{6}{s+1} = \frac{(5-s)(s+2)}{s+1}$

$\therefore \quad \mathcal{L}\{x\} = \frac{(5-s)(s+2)}{(s^2-4)(s+1)} = \frac{5-s}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$

$\therefore \quad 5 - s = A(s+1) + B(s-2).$

Putting $s = -1$ gives us $5-(-1) = B(-1-2) \Rightarrow B = -2$

Putting $s = 2$ gives us $5 - 2 = A(2+1) \Rightarrow A = 1.$

$\therefore \mathcal{L}\{x\} = \frac{A}{s-2} + \frac{B}{s+1} = \frac{1}{s-2} - \frac{2}{s+1}.$

$\therefore \quad x(t) = \mathcal{L}^{-1}\{\frac{1}{s-2}\} - 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} = e^{2t} - 2e^{-t}.$

From the original equations, $y(t) = [x'(t) - 2]/2$

$= [2e^{2t} + 2e^{-t} - 2]/2 = e^{2t} + e^{-t} - 1.$

§4. Convolutions and the inverse of the Laplace transform

We have seen that the Laplace transform was very useful in solving linear constant coeff. ODEs. In the process we used, we had to find the inverse of the Laplace transform to get our final answers. This involved manipulating the functions we got for $\mathcal{L}\{x\}$ or $\mathcal{L}\{y\}$ into a form we recognized as the Laplace transform of certain functions. For example, if we got $\frac{3}{s-2}$, then $\mathcal{L}^{-1}\{\frac{3}{s-2}\}(t) = 3.\, e^{2t}$.

We shall use the convolution of two functions as an aid to finding inverse Laplace transforms.

<u>Def</u> Let f and g be piecewise-continuous functions on $[0,\infty)$ which are of exponential order. We define the <u>convolution</u> of f and g to be the function

$$(f*g)(t) = \int_0^t f(u).\, g(t-u)\, du.$$

<u>Prop 4.</u> $(g*f)(t) = (f*g)(t)$

<u>Proof</u> We have $(g*f)t = \int_0^t g(u).\, f(t-u)\, du$

Put $u = t-v$.
Then $du = -dv$
$u = 0 \Rightarrow v = t-u = t$
$u = t \Rightarrow v = t-t = 0$
(change of variable)

$$= \int_t^0 g(t-v).\, f(v)(-dv)$$

$$= -\int_0^t - f(v).\, g(t-v).\, dv$$

$$= \int_0^t f(v).\, g(t-v)\, dv$$

$$= \int_0^t f(u).\, g(t-u)\, du = (f*g)(t)$$

because v is just a dummy variable and can be replaced by any other dummy variable.

Ex. 2 Let $f(t) = t^2$ and $g(t) = t$. Find $(f*g)(t)$

Sol. $$(f*g)(t) = \int_0^t f(u) \cdot g(t-u)\, du$$
$$= \int_0^t u^2 \cdot (t-u) \cdot du$$
$$= \int_0^t \left[t \cdot u^2 - u^3\right] du$$
$$= \left[t \cdot \frac{u^3}{3} - \frac{u^4}{4}\right]_0^t = t \cdot \frac{t^3}{3} - \frac{t^4}{4} = \frac{t^4}{12}.$$

Ex. 3 Let $f(t) = t$ & $g(t) = 1$. Find $(f*g)(t)$

Sol. $$(f*g) = \int_0^t f(u) \cdot g(t-u) \cdot du = \int_0^t u \cdot 1 \cdot du$$
$$= \left[\frac{u^2}{2}\right]_0^t = \frac{t^2}{2}.$$

Ex. 4 Let $f(t) = e^{at}$ & $g(t) = e^{bt}$. Find $(f*g)(t)$.

Sol. $$(f*g)(t) = \int_0^t f(u) \cdot g(t-u) \cdot du = \int_0^t e^{au} \cdot e^{b(t-u)} \cdot du$$
$$= \int_0^t e^{bt} \cdot e^{(a-b)u} \cdot du = \frac{e^{bt}}{a-b} \left[e^{(a-b)t}\right]_0^t$$
$$= \frac{e^{bt}}{a-b} \cdot \left(e^{(a-b)t} - 1\right) = \frac{e^{at} - e^{bt}}{a-b}$$
provided $a \neq b$.

If $a = b$, then $$(f*g)(t) = \int_0^t e^{bt} \cdot 1 \cdot du$$
$$= e^{bt} \cdot [u]_0^t = te^{bt} = te^{at}.$$

Notice that if we treat "a" as a variable and let $a \to b$ (and treat b & t as constants),
$$\lim_{a \to b} \frac{e^{at} - e^{bt}}{a-b} = \lim_{a \to b} \frac{te^{at} - 0}{1 - 0} = te^{at}$$ by l'Hôpital's Rule.

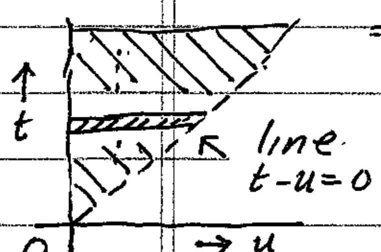
Theorem 5. Let $f(t)$ & $g(t)$ be piecewise-continuous functions of exponential order. Then

(a) $\mathcal{L}\{(f*g)(t)\}(s) = \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s)$.

(b) $\mathcal{L}^{-1}\{F(s) \cdot G(s)\}(t) = \left(\mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}\right)(t)$.

Sketch of the proof: We have

(a) $\mathcal{L}\{(f*g)(t)\}(s) = \int_0^\infty e^{-st} \cdot (f*g)(t)\,dt$

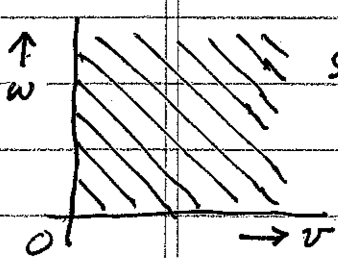


$$= \int_0^\infty e^{-st}\left[\int_0^t f(u) \cdot g(t-u) \cdot du\right] \cdot dt$$

$$= \iint_{R_1} e^{-st} \cdot f(u) \cdot g(t-u) \cdot du\,dt$$

$$= \iint_{R_2} e^{-s(v+w)} \cdot f(v) \cdot g(w) \cdot dv\,dw$$

Put $v = u$ & $w = t-u$
$= t - v$
So $t = v + w$



$$= \int_0^\infty \int_0^\infty e^{-s(v+w)} \cdot f(v) \cdot g(w)\,dv\,dw$$

$$= \left(\int_0^\infty e^{-sv} \cdot f(v)\,dv\right)\left(\int_0^\infty e^{-sw} \cdot g(w)\,dw\right)$$

$$= \left(\int_0^\infty e^{-st} f(t)\,dt\right)\left(\int_0^\infty e^{-st} g(t)\,dt\right)$$

because v & w are dummy variables

$$= \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s)$$

(b) Part (b) follows immediately from part (a) by letting $F(s) = \mathcal{L}\{f(t)\}(s)$ & $G(s) = \mathcal{L}\{g(t)\}(t)$ and by taking the inverse Laplace transform of both sides of (a).

Ex. 5 Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} \cdot \frac{1}{s+2}\right\}(t)$

Sol. $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)} \cdot \frac{1}{(s+2)}\right\}(t) = \left(\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}\right)(t)$

$$= e^t * e^{-2t} = \int_0^t e^u \cdot e^{-2(t-u)} \cdot du = \int_0^t e^{3u} \cdot e^{-2t}\,du$$

$$= \frac{1}{3} e^{-2t}\left[e^{3u}\right]_0^t = \frac{1}{3} e^{-2t} \cdot (e^{3t} - 1) = \frac{1}{3}(e^t - e^{-2t}).$$

Ex. 6. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+1}\right\}$

Sol. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+1}\right\}(t) = \left(\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\right)(t)$

$= t * e^{t} = \int_0^t u \cdot e^{-(t-u)} \cdot du = \int_0^t e^{-t} \cdot u e^{u} du$

$= e^{-t}\left[(u-1).e^{u}\right]_0^t = e^{-t}\left[(t-1)e^{t} - (-1).e^{0}\right]$

$= (t-1) + e^{-t}.$

Ex. 7 Find $\mathcal{L}^{-1}\left\{\frac{2}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{2}{s^2+4}\right\}$

Sol. $\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{2}{s^2+4}\right\}(t) = \left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}\right)(t)$

$= 1 * \sin(2t) = \int_0^t 1 \,.\, \sin[2(t-u)]\, du = \dots$

Now this integral looks a little complicated, so let us use a trick from Prop. 1

$= \sin(2t) * 1 = \int_0^t \sin(2u) \,.\, 1 \,.\, du$

$= \left[-\frac{1}{2}\cos(2u)\right]_0^t = \frac{1}{2}\left[1 - \cos(2t)\right] = \sin^2(t).$

Ex. 8 Find $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\cdot\frac{s}{s^2+1}\right\}$

Sol. $\mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\cdot\frac{s}{s^2+1}\right\}(t) = \left(\mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}\right)(t)$

$= 2\sin t * \cos t = \int_0^t 2\sin(u) \,.\, \cos(t-u) \,.\, du$

$= 2\int_0^t \sin u\left[\cos t \cos u + \sin t . \sin u\right] = \int_0^t \left[\cos t . \sin(2u) + \sin t \,.\, 2\sin^2 u\right] du$

$= \cos(t)\int_0^t \sin(2u)\, du + \sin(t)\int_0^t \left[1 - \cos(2u)\right] du$

$= \cos t . \left[\frac{1}{2}\cos(2u)\right]_0^t + \sin(t) . \left[t + \frac{1}{2} . \sin(2u)\right]_0^t$

$= t . \sin t + \frac{1}{2}\cos t\left[\cos(2t) - 1\right] + \sin t \,.\, \frac{1}{2} \,.\, 2\sin(2t)$

$= t \,.\, \sin t + \frac{1}{2}\cos(t) . \left[\cos^2(t) + \sin^2(t)\right] - \frac{1}{2}\cos(t) = t . \sin(t).$

§5. Solving ODEs with discontinuous R.H.S. terms

We shall use the Laplace transform to solve linear constant coeff non-homogeneous ODEs in which the RHS are distinuous step functions. But for this we first need to find the Laplace transforms of step functions. Recall that the unit-jump function at a was defined by $u_a(t) = \begin{cases} 0 & \text{if } 0 \le t < a, \\ 1 & \text{if } t \ge a. \end{cases}$

Ex.1 Find $\mathcal{L}\{u_a(t)\}$ for $a \ge 0$.

Sol. $\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st} \cdot u_a(t) \cdot dt$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt$$

$$= 0 + \lim_{R\to\infty} \int_a^R e^{-st} dt = \lim_{R\to\infty} \left[-\frac{e^{-st}}{s} \right]_a^R$$

$$= \frac{e^{-as}}{s} - \lim_{R\to\infty} \frac{e^{-Rs}}{s} = \frac{e^{-as}}{s} - 0 = \frac{e^{-as}}{s}.$$

Ex.2 Let $f(t) = \begin{cases} 3 & 2 \le t < 5, \\ 0 & \text{otherwise} \end{cases}$. Find $\mathcal{L}\{f(t)\}$

Sol.

We have $f(t) = 3u_2(t) - 3u_5(t)$. So

$$\mathcal{L}\{f(t)\} = 3\left[\mathcal{L}\{u_2(t)\} - \mathcal{L}\{u_5(t)\} \right]$$

$$= 3 \cdot \left[\frac{e^{-2s}}{s} - \frac{e^{-5s}}{s} \right] = \frac{3}{s} \cdot \left(e^{-2s} - e^{-5s}\right).$$

Prop. 6 $\mathcal{L}\{u_a(t) \cdot f(t-a)\}(s) = e^{-as} \cdot \mathcal{L}\{f(t)\}(s)$

Proof: $u_a(t) \cdot f(t-a) = \begin{cases} 0 & \text{if } 0 \le t < a, \\ f(t-a) & \text{if } t \ge a. \end{cases}$

So $\mathcal{L}\{u_a(t) \cdot f(t-a)\}(s) = \int_0^\infty e^{-st} \cdot u_a(t) \cdot f(t-a) \cdot dt$

$$= \int_0^a e^{-st}.0.dt + \int_a^\infty e^{-st}.1.f(t-a).dt$$

Put $w = t-a$, $dw = dt$, $t=0 \Rightarrow w = 0$, $t\to\infty \Rightarrow w\to\infty$

$$= \int_0^\infty e^{-s(w+a)}.f(w).dw = e^{-as}\int_0^\infty e^{-ws}.f(w)dw$$

$$= e^{-as}\int_0^\infty e^{-st}f(t).dt = e^{-as}.\mathcal{L}\{f(t)\}(s).$$

Ex. 3 Let $g(t) = \begin{cases} 0 & \text{if } 0\leq t<3, \\ 2t+1 & \text{if } t\geq 3. \end{cases}$ Find $\mathcal{L}\{g(t)\}$.

Sol

We have $g(t) = \begin{cases} 0 & \text{if } 0\leq t<3, \\ 2(t-3)+7 & \text{if } t\geq 3. \end{cases}$

$\therefore g(t) = u_3(t).f(t-3)$ where $f(t) = 2t+7$.

$$\therefore \mathcal{L}\{g(t)\} = \mathcal{L}\{u_3(t).f(t-3)\}$$

$$= e^{-3s}.\mathcal{L}\{f(t)\} = e^{-3s}\mathcal{L}\{2t+7\}$$

$$= e^{-3s}\left(\mathcal{L}\{2t\}+\mathcal{L}\{7\}\right) = e^{-3s}\left(\frac{2}{s^2}+\frac{7}{s}\right).$$

Ex. 4 Find $\mathcal{L}^{-1}\{e^{-3s}.\frac{2}{s^3}\}(t)$.

Sol $\mathcal{L}^{-1}\{e^{-3s}.\frac{2}{s^3}\}(t) = \mathcal{L}^{-1}\{e^{-3s}.\mathcal{L}\{f(t)\}\}(t)$ where $\mathcal{L}\{f(t)\} = \frac{2}{s^3}$

$= u_3(t).f(t-3)$ so $f(t) = t^2$

$= u_3(t).(t-3)^2$

$= \begin{cases} 0 & \text{if } 0\leq t<3, \\ (t-3)^2 & \text{if } t\geq 3. \end{cases}$

Ex.5 Find $\mathcal{L}^{-1}\{e^{-\pi s/2}.\frac{s}{s^2+4}\}$.

Sol. $\mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}(s)}.\frac{s}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{e^{-\frac{\pi}{2}.s}.\mathcal{L}\{f(t)\}\right\}$, $\mathcal{L}\{f(t)\} = \frac{s}{s^2+4}$

$= u_{\pi/2}(t).f(t-\pi/2) = u_{\pi/2}(t).\cos(t-\pi/2)$ $f(t) = \cos(2t)$

$= u_{\pi/2}(t).[\underbrace{\cos(t)\cos(\pi/2)}_{=0} + \sin(t).\underbrace{\sin(\pi/2)}_{=1}] = \begin{cases} 0 & \text{if } t<\pi/2 \\ \sin(t) & \text{if } t\geq\frac{\pi}{2}. \end{cases}$

Ex. 6 Find the solution of the ODE

$$y'(t) + 2y(t) = 2 - 2U_3(t) \quad \text{with } y(0) = 4.$$

Sol. $\mathcal{L}\{y'(t)\} + 2\,\mathcal{L}\{y(t)\} = \mathcal{L}\{2\} - 2\,\mathcal{L}\{U_3(t)\}$

$\therefore \quad [s.\,\mathcal{L}\{y\} - y(0)] + 2\,\mathcal{L}\{y\} = \frac{2}{s} - \frac{2e^{-3s}}{s}$

$\therefore \quad (s+2)\,\mathcal{L}\{y\} = y(0) + \frac{2}{s} - \frac{2e^{-3s}}{s} = \frac{4s+2}{s} - \frac{2e^{-3s}}{s}$

$\therefore \quad \mathcal{L}\{y\} = \frac{4s+2}{s(s+2)} - \frac{2e^{-3s}}{s(s+2)}$

$\frac{4s+2}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \implies 4s+2 = A(s+2) + B.s$

Putting $s=0$, gives $2 = A(0+2) \Rightarrow A = 1$

Putting $s=-2$, gives $-8+2 = B(-2) \Rightarrow B = 3$

$\frac{2}{s(s+2)} = \frac{C}{s} + \frac{D}{s+2} \implies 2 = C(s+2) + D.s$

Putting $s=0$ gives $2 = C(0+2) \Rightarrow C = 1$

Putting $s=-2$ gives $2 = D(-2) \Rightarrow D = -1$

$\therefore \quad \mathcal{L}\{y\} = \left(\frac{A}{s} + \frac{B}{s+2}\right) - e^{-3s}\left(\frac{C}{s} + \frac{D}{s+2}\right)$

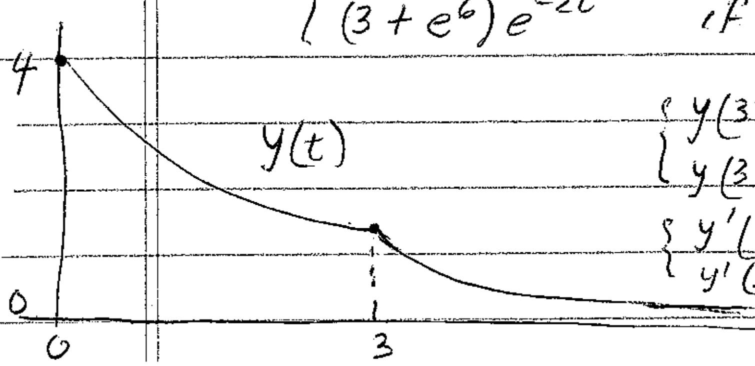
$= \left(\frac{1}{s} + \frac{3}{s+2}\right) - e^{-3s}\left(\frac{1}{s} - \frac{1}{s+2}\right)$

$\therefore \quad y(t) = (1+3e^{-2t}) - U_3(t)\, f(t-3)$ where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+2}\right\} = 1 - e^{-2t}$

$= (1+3e^{-2t}) - U_3(t).\,(1 - e^{-2(t-3)})$

$= \begin{cases} (1+3e^{-2t}) - 0 & \text{if } 0 \le t < 3 \\ (1+3e^{-2t}) - (1+e^{-2t}.e^{6}) & \text{if } t \geq 3 \end{cases}$

$= \begin{cases} 1+3e^{-2t} & \text{if } 0 \le t < 3 \\ (3+e^{6})e^{-2t} & \text{if } t \geq 3 \end{cases}$



$\begin{cases} y(3^-) = 1 + 3e^{-6}, \\ y(3^+) = 3e^{-6} + 1. \end{cases}$

$\begin{cases} y'(3^-) = (-2)[0 + 3e^{-6}], \\ y'(3^+) = (-2)[1 + 3e^{-6}]. \end{cases}$

END

Ex. 7 Find the solution of the ODE

$$y'(t) - y(t) = 3 + 2u_1(t) \quad \text{with } y(0) = 1$$

Sol. $\mathcal{L}\{y'(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{3\} + 2\,\mathcal{L}\{u_1(t)\}$

$\therefore\ [s\,\mathcal{L}\{y\} - y(0)] - \mathcal{L}\{y\} = \frac{3}{s} + 2\cdot\frac{e^{-s}}{s}$

$\therefore\ (s-1)\cdot\mathcal{L}\{y\} = 1 + \frac{3}{s} + 2\frac{e^{-s}}{s} = \frac{3s+1}{s} + \frac{2e^{-s}}{s}$

$\therefore\ \mathcal{L}\{y\} = \frac{3s+1}{s(s-1)} + \frac{2e^{-s}}{s(s-1)}$.

$\frac{3s+1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} \Rightarrow 3s+1 = A(s-1) + Bs$

Putting $s=0$ gives us $3(0)+1 = A(0-1) \Rightarrow A = -1$

Putting $s=1$ gives us $3(1)+1 = B(1) \Rightarrow B = 4$

$\frac{2}{s(s-1)} = \frac{C}{s} + \frac{D}{s-1} \Rightarrow 2 = C(s-1) + Ds$

Putting $s=0$ gives us $2 = C(-1) \Rightarrow C = -2$

Putting $s=1$ gives us $2 = D(1) \Rightarrow D = 2$

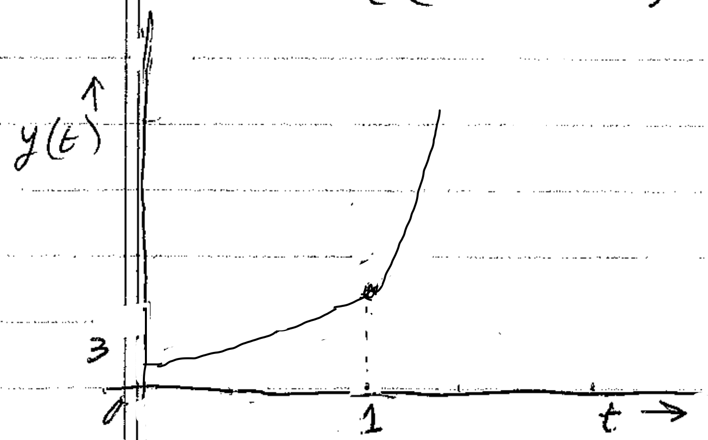
$\therefore\ \mathcal{L}\{y\} = \left(\frac{4}{s-1} - \frac{1}{s}\right) + e^{-s}\left(\frac{2}{s-1} - \frac{2}{s}\right)$

$\therefore\ y(t) = \mathcal{L}^{-1}\left\{\frac{4}{s-1} - \frac{1}{s}\right\} + u_1(t)\cdot f(t-1)$ where $\mathcal{L}\{f\} = \frac{2}{s-1} - \frac{2}{s}$

$= (4e^t - 1) + u_1(t)(2e^{t-1} - 2)$ so $f(t) = 2e^t - 2$

$$= \begin{cases} 4e^t - 1 & \text{if } 0 \le t < 1, \\ (4e^t - 1) + (2e^{t-1} - 2) & \text{if } t \ge 1. \end{cases}$$

$$= \begin{cases} 4e^t - 1 & \text{if } 0 \le t < 1, \\ (4 + 2e^{-1})e^t - 3 & \text{if } t \ge 1. \end{cases}$$



$$\begin{cases} y(1-) = 4e - 1 \\ y(1+) = 4e - 1 \end{cases}$$

$$\begin{cases} y'(1^-) = 4e \\ y'(1^+) = 4e + 2e^{-1} \end{cases}$$