Ch. 6 - Variable coefficients Linear ODE

§1. Series solutions of ODEs about analytic points

In this chapter we shall concentrate on second order homog. linear ODEs with variable coefficients. The same method used for 2nd order ODEs can be used for higher order ODEs but the calculations become cumbersome. Recall that a linear 2nd order ODE was an ODE that can be written in the form

$$y'' + P(x).y' + Q(x).y = 0 \qquad (*)$$

Def. A function f is analytic at x_0 if the Taylor series of f about x_0, $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$ exists and converges to $f(x)$ for all x in some open interval (a,b) which contains x_0.

Def. The point x_0 is called an analytic (or an ordinary) point of the ODE

$$y'' + P(x).y' + Q(x).y = 0 \qquad (*)$$

if $P(x)$ & $Q(x)$ are both analytic at x_0.

The point x_0 is a singular point of $(*)$ if at least one of the two functions $P(x)$ & $Q(x)$ is not analytic at x_0.

Ex. 1 (a) The point $x_0 = 0$ is an analytic point of the ODE $y'' + 3x.y' + \frac{1}{1-x^2} y = 0$.

(b) The point $x_0 = 1$ is a singular point of the ODE $y'' + 3x.y' - \frac{1}{1-x^2}.y = 0$.

(c) The point $x_0 = 0$ is a singular point of the ODE

$$x^2.y'' - 2x.y' + 4y = 0.$$

For a first-order linear ODE the definitions are similar. So the point $x_0 = 0$ is an analytic point of the ODE, $y' - 2x.y = 0$. We are guaranteed to have a non-trivial power-series solution about an analytic point of a first-order linear ODE; and two linearly independent power-series solutions about an analytic point of a 2nd order linear ODE.

Theorem 1 Let x_0 be an analytic point of the ODE

$$y'' + P(x).y' + Q(x).y = 0 \qquad (*).$$

Then (*) has two linearly independent power-series solutions of the form $y = \sum_{n=0}^{\infty} c_n.(x-x_0)^n$. These two power-series are guaranteed to be convergent in the interval (x_0-R, x_0+R) where $R = \min\{R_1, R_2\}$ and

R_1 = radius of convergence of the Taylor-series of $P(x)$ about x_0

& R_2 = radius of convergence of the Taylor-series of $Q(x)$ about x_0.

Ex. 2 Find two linearly indep. power-series solutions of the Airy ODE (*): $y'' - x.y = 0$ about the point $x_0 = 0$, and give the first three non-zero terms and the general term of each of your power-series solutions.

Sol. Here $P(x) = 0$ & $Q(x) = x$. So the Taylor-series of $P(x)$ & $Q(x)$ about $x_0 = 0$ will be $P(x) = 0 + 0.x + 0.x^2 + \dots$ and $Q(x) = 0 + 1.x + 0.x^2 + \dots$. So $R_1 = +\infty$ & $R_2 = +\infty$.

Hence our solution is guaranteed to converge on $(-\infty, \infty)$.

Now suppose $y(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n . x^n + \dots$

Then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1} x^n + \dots$

and $y''(x) = 1(2)a_2 + 2(3)a_3 x + 3(4)a_4 x^2 + \dots + (n+1)(n+2)a_{n+2} x^n + \dots$

So our ODE $y'' - xy \equiv 0$ becomes

$$2a_2 + 6a_3 x + 12a_4 x^2 + \cdots + (n+1)(n+2)a_{n+2} \cdot x^n + \cdots$$
$$- a_0 x - a_1 x^2 - \cdots - a_{n-1} \cdot x^n - \cdots \equiv 0$$

$$\therefore 2a_2 + (6a_3 - a_0)x + (12a_4 - a_1)x^2 + \cdots + [(n+1)(n+2)a_{n+2} - a_{n-1}]x^n + \cdots \equiv 0$$

$\therefore 2a = 0$, $(6a_3 - a_0) = 0$, $(12a_4 - a_1) = 0$, and in general
$(n+1)(n+2)a_{n+2} - a_{n-1} = 0 \Rightarrow a_{n+2} = (a_{n-1})/(n+1)(n+2)$.

$\therefore a_0 = \text{arb.}$, $a_1 = \text{arb.}$, $a_2 = 0$, ..., and in general
$a_{n+2} = (a_{n-1})/(n+1)(n+2)$.

Putting $n=1$ gives us $a_3 = a_0/(2)(3)$
" $n=2$ gives us $a_4 = a_1/(3)(4)$
" $n=3$ gives us $a_5 = a_2/(4)(5) = 0$
" $n=4$ " $a_6 = a_3/(5)(6) = a_0/(2)(3)(5)(6)$
$n=5$ " $a_7 = a_4/(6)(7) = a_1/(3)(4)(6)(7)$

$$\therefore y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_7 x^7 + \cdots + a_n x^n + \cdots$$
$$= a_0 \left[1 + \frac{x^3}{(2)(3)} + \frac{x^6}{(2)(3)(5)(6)} + \cdots + \frac{x^{3n}}{2(3)(5)(6)\cdots(3n-1)(3n)} + \cdots\right]$$
$$+ a_1 \left[x + \frac{x^4}{(3)(4)} + \frac{x^7}{(3)(4)(6)(7)} + \cdots + \frac{x^{3n+1}}{(3)(4)(6)(7)\cdots(3n)(3n+1)} + \cdots\right]$$

because $a_{3n+2} = 0$ for all $n \geq 0$.

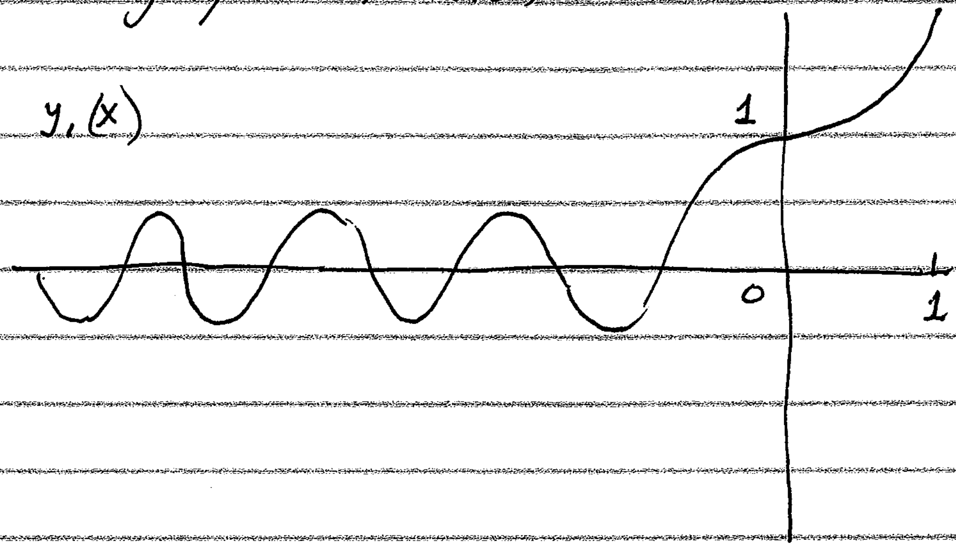
So $$y_1(x) = 1 + \frac{x^3}{2(3)} + \frac{x^6}{2(3)(5)(6)} + \cdots + \frac{x^{3n}}{2(3)(5)(6)\cdots(3n-1)(3n)} + \cdots$$

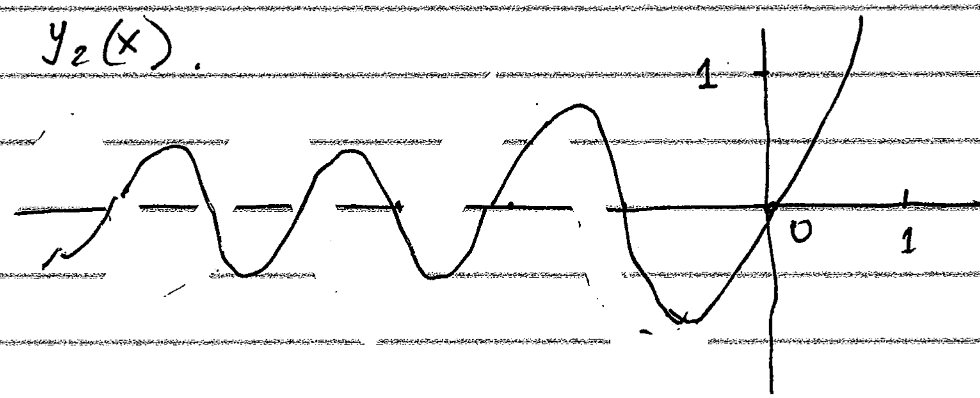
and $$y_2(x) = x + \frac{x^4}{3(4)} + \frac{x^7}{3(4)(6)(7)} + \cdots + \frac{x^{3n+1}}{3(4)(6)(7)\cdots(3n)(3n+1)} + \cdots$$

will be two linearly independent solutions.

Naturally these two functions $y_1(x)$ & $y_2(x)$ are called Airy's functions in honor of their discoverer. The general solution of our ODE (*) will be $y = C_1 \cdot y_1(x) + C_2 \cdot y_2(x)$.

The functions $y_1(x)$ & $y_2(x)$ are a bit similar to $\cos(x)$ & $\sin(x)$ when x is negative and a bit similar to $\cosh(x)$ & $\sinh(x)$ when x is positive. By the way $\cosh(x) = (e^x + e^{-x})/2$ & $\sinh(x) = (e^x - e^{-x})/2$. Below are the graphs of $y_1(x)$ & $y_2(x)$.





Ex 3 Find two linearly indep. power-series solutions of the ODE (*) $y'' - (1+x)y \equiv 0$ about the point $x_0 = 0$. Give the first 5 non-zero terms of your power-series solutions

Sol. Again, we can easily see that x_0 is an analytic point of the ODE (*) and we are guanteed two linearly indep. power series solution which converge in $(-\infty, \infty)$.

Let $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n \cdot x^n + \cdots$

Then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1) \cdot a_{n+1} x^n + \cdots$

& $y''(x) = 2a_2 + 2(3)x + 3(4)x^2 + \cdots + (n+1)(n+2)\, a_{n+2} x^n + \cdots$

So $y'' - (1+x) \cdot y = y'' - y - xy \equiv 0$ becomes

$$\begin{aligned} &2a_2 + 2(3)a_3 x + 3(4)a_4 x^3 + \cdots + (n+1)(n+2)a_{n+2} x^n + \cdots \\ &- a_0 - a_1 x - a_2 x^2 - \cdots - a_n \cdot x^n - \cdots \\ &\quad - a_0 \cdot x - a_1 \cdot x^2 - \cdots - a_{n-1} \cdot x^n - \cdots \equiv 0 \end{aligned}$$

$$\therefore (2a_2 - a_0) + [2(3)a_3 - a_1 - a_0]\,x + [3(4)a_4 - a_2 - a_1]\,x^2 + \cdots + [(n+1)(n+2)\,a_{n+2} - a_n - a_{n-1}] \cdot x^n + \cdots \equiv 0$$

$\therefore 2a_2 - a_0 = 0$ and $(n+1)(n+2)\,a_{n+2} - a_n - a_{n-1} = 0$

$\therefore a_2 = \dfrac{a_0}{2} = \dfrac{a_0}{1(2)}$ and $a_{n+2} = \dfrac{a_{n-1} + a_n}{(n+1)(n+2)}$

Putting $n=1$ gives $a_3 = \dfrac{a_0}{2(3)} + \dfrac{a_1}{2(3)}$

Putting $n=2$ gives $a_4 = \dfrac{a_1}{3(4)} + \dfrac{a_2}{3(4)} = \dfrac{a_0}{2(3)(4)} + \dfrac{a_1}{3(4)}$

Putting $n=3$ gives $a_5 = \dfrac{a_2}{4(5)} + \dfrac{a_3}{4(5)} = \dfrac{a_0}{2(4)(5)} + \dfrac{a_0}{2(3)(4)5} + \dfrac{a_1}{2(3)(4)(5)}$

$$= \frac{a_0}{2(3)(4)(5)}(3+1) + \frac{a_1}{2(3)(4)(5)}.$$

$$\begin{aligned} \therefore y(x) &= a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 + a_4 \cdot x^4 + a_5 \cdot x^5 + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{1(2)} x^2 + \frac{a_0 + a_1}{2(3)} \cdot x^3 + \frac{a_0 + 2a_1}{2(3)(4)} x^4 + \frac{4a_0 + a_1}{2(3)(4)(5)} \cdot x^5 + \cdots \\ &= a_0\left[1 + 0 \cdot x + \frac{x^2}{1(2)} + \frac{x^3}{2(3)} + \frac{x^4}{2(3)(4)} + \frac{4x^5}{2(3)(4)(5)} + \cdots\right] \\ &\quad + a_1\left[0 + 1 \cdot x + \frac{x^2}{1(2)} + \frac{x^3}{2(3)} + \frac{2x^4}{2(3)(4)} + \frac{x^5}{2(3)(4)(5)} + \cdots\right] \end{aligned}$$

$\therefore y_1(x) = \left[1 + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{4x^5}{5!} + \cdots\right]$ and

$y_2(x) = \left[\dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{2x^4}{4!} + \dfrac{x^5}{5!} + \cdots\right].$

Ex. 4 Find the first 6 non-zero terms of the power-series solution of the ODE (*) $y'' - y' + xy \equiv 0$ about $x_0 = 0$, with $y(0) = 4$ and $y'(0) = 6$.

Sol. Let $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n \cdot x^n + \cdots$

Then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1} x^n + \cdots$

& $y''(x) = 2a_2 + 2(3)a_3 \cdot x + 3(4)a_4 x^2 + \cdots + (n+1)(n+2)a_{n+2} \cdot x^n + \cdots$

So $y(0) = a_0$ and $y'(0) = a_1$. $\therefore$ $a_0 = 4$ and $a_1 = 6$

Also (*) becomes

$$\begin{aligned} &2a_2 + 2(3)a_3 x + 3(4)a_4 x^2 + \cdots + (n+1)(n+2)a_{n+2} \cdot x^n + \cdots \\ &- a_1 - 2a_2 \cdot x - 3a_3 \cdot x^2 - \cdots - (n+1) \cdot a_{n+1} \cdot x^n - \cdots \\ &+ a_0 \cdot x + a_1 \cdot x^2 + \cdots + a_{n-1} x^n + \cdots \equiv 0. \end{aligned}$$

$$\therefore (2a_2 - a_1) + [2(3)a_3 - 2a_2 + a_0] \cdot x + [3(4)a_4 - 3a_3 + a_1]x^2 + \cdots + [(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + a_{n-1}]x^n + \cdots \equiv 0$$

$\therefore$ $2a_2 - a_1 = 0$ & $(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + a_{n-1} = 0$

$\therefore$ $a_2 = \dfrac{a_1}{2} = 3$ & $a_{n+2} = \dfrac{(n+1)a_{n+1} - a_{n-1}}{(n+1)(n+2)}$ for $n \geq 1$

Putting $n=1$ gives $a_3 = \dfrac{(1+1)a_2 - a_0}{2(3)} = \dfrac{2(3) - 4}{6} = \dfrac{1}{3}$

Putting $n=2$ gives $a_4 = \dfrac{(2+1)a_3 - a_1}{3(4)} = \dfrac{1-6}{12} = \dfrac{-5}{12}$

Putting $n=3$ gives $a_5 = \dfrac{(3+1)a_4 - a_2}{4(5)} = \dfrac{-5/3 - 3}{20} = \dfrac{-14}{60} = \dfrac{-7}{30}$

$$\therefore y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$
$$= 4 + 6x + 3x^2 + \frac{x^3}{3} - \frac{5x^4}{12} - \frac{7x^5}{30} + \cdots$$

and we have the first 6 non-zero terms.

§2. Cauchy-Euler (equi-dimensional) ODEs.

The 2nd-order lin ODEs with variable coefficients behave very nicely at the analytic points of the ODE. At a singular point, we are not guaranteed two linearly indep. series. However at the "nice" (regular) singular points we are guaranteed two linearly independent solutions which are almost like power-series. So for the rest of this chapter we will deal only with regular singular points.

Def. The point x_0 is a regular singular point of the ODE
$$y'' + P_1(x).y' + P_0(x)y = 0 \qquad (*),$$
if x_0 is a singular point of $(*)$ and if $(x-x_0)P_1(x)$ and $(x-x_0)^2 P_0(x)$ are analytic functions.

Ex. 1 (a) The point $x_0 = 0$ is a regular singular point of the ODE $y'' - \frac{3}{x}y' + \frac{5}{x^2}y = 0$.

(b) The point $x_0 = 1$ is a regular singular point of the ODE $y'' + \frac{x}{x^2-1}y' + \frac{2x^3}{(x-1)^2}y = 0$.

(c) The point $x_0 = 0$ is a regular singular point of the ODE $y'' + \frac{e^x}{x}.y' + \frac{\cos x}{x^2}y = 0$.

Def. A Cauchy-Euler (or equidimensional) ODE is one of the form
$$a_n . x^n . y^{(n)} + a_{n-1}.x^{n-1}.y^{(n-1)} + \cdots + a_1.x.y' + a_0.y = 0$$
where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$.

When $n=2$, we can write this ODE as

$$a.x^2y'' + b.x.y' + c.y = 0 \qquad (*)$$

Notice that if we divide throughout by ax^2, we get

$$y'' + \frac{b}{a}.\frac{1}{x}.y' + \frac{c}{a}.\frac{1}{x^2}y = 0.$$

So $x_0 = 0$ is a regular singular point of this ODE.

Proposition 2: The tranformation $x = e^t$ reduces the ODE

$$a_n.x^n y^{(n)} + a_{n-1}.x^{n-1}.y^{(n-1)} + \cdots + a_1.x.y' + a_0.y = 0 \qquad (*)$$

into a linear constant coeff. ODE in t and y.

Sketch of proof: Let $D = \frac{d}{dx}$ and $\Delta = \frac{d}{dt}$. Then $(*)$ can be written as

$$(a_n.x^nD^n + a_{n-1}x^{n-1}D^{n-1} + \cdots + a_1D + a_0)y = 0. \qquad (**)$$

Now $(xD)y = x\frac{dy}{dx} = x.\frac{dy}{dt}.\frac{dt}{dx}$

$$= x.\frac{dy}{dt}.\frac{1}{x} = \frac{dy}{dt} = (\Delta)y$$

$x = e^t$

$t = \ln x$

$\frac{dt}{dx} = \frac{1}{x}$

$\therefore\ xD = \Delta$. So

$$\Delta^2 = (xD)(x.D) = x.[D(x.D)]$$
$$= x.[1.D + x.D^2]$$
$$= xD + x^2D^2 = \Delta + x^2D^2$$

$$\therefore\ x^2D^2 = \Delta^2 - \Delta = \Delta(\Delta - 1)$$

Also $\Delta^3 = (xD)[(xD)(xD)] = xD[xD + x^2D^2]$

$$= x\{1.D + x.D^2 + 2x.D^2 + x^2.D^3]$$
$$= xD + 3x^2D^2 + x^3D^3$$
$$= \Delta + 3\Delta(\Delta-1) + x^3D^3$$

$$\therefore\ x^3D^3 = \Delta^3 + 3\Delta^2 - 3\Delta + \Delta = \Delta(\Delta-1)(\Delta-2)$$

Similarly $x^nD^n = \Delta(\Delta-1)(\Delta-2)\cdots(\Delta-(n-1))$ by using Mathematical Induction.

Hence (**) become

$$[a_n \cdot \Delta(\Delta-1)\cdots(\Delta-(n-1)) + \cdots + a_2 \cdot \Delta(\Delta-1) + a_1 \cdot \Delta + a_0]\, y = 0$$

which is a linear constant coefficient ODE in t & y.

Ex. 1 Find the general solution of the ODE

$$x^2 y'' - 4x \cdot y' + 6y = 0 \qquad (*)$$

Sol Let $x = e^t$. Then $xD = \Delta$ and $x^2 D^2 = \Delta(\Delta-1)$.

Now $(*)$ can be rewritten as

$$[x^2D^2 - 4xD + 6]\, y = 0$$

and this therefore becomes

$$[\Delta(\Delta-1) - 4\Delta + 6]\, y = 0$$

$$\therefore\ [\Delta^2 - \Delta - 4\Delta + 6]\, y = 0$$

$$\therefore\ (\Delta^2 - 5\Delta + 6)\, y = 0$$

Aux. eq. is $\Delta^2 - 5\Delta + 6 = 0 \Rightarrow (\Delta-2)(\Delta-3) = 0$

$\therefore\ \Delta = 2$ or 3. $\therefore\ y = C_1 e^{2t} + C_2 e^{3t}$

$$= C_1 (e^t)^2 + C_2 (e^t)^3$$

$$= C_1 \cdot x^2 + C_2 \cdot x^3$$

is the general solution.

Ex. 2 Find the general solution of the ODE

$$x^2 y'' + 2x \cdot y' - 2y = 12x^2, \qquad (*).$$

Sol. Again let $x = e^t$. Then $xD = \Delta$ & $x^2D^2 = \Delta(\Delta-1)$.

Now $(*)$ is $[x^2D^2 + 2x \cdot D - 2]\, y = 12x^2$.

$$\therefore\ [\Delta(\Delta-1) + 2\Delta - 2]\, y = 12 \cdot (e^t)^2$$

$$\therefore\ [\Delta^2 - \Delta + 2\Delta - 2]\, y = 12 \cdot e^{2t} \qquad (**)$$

$\therefore\ (\Delta^2 + \Delta - 2)\, y = 0$ is the homog. equation

$\therefore\ (\Delta - 1)(\Delta + 2) = 0 \Rightarrow \Delta = 1$ or -2.

$$\therefore\ y_c = C_1 e^t + C_2 \cdot e^{-2t} = C_1 \cdot x + C_2 \cdot x^{-2}.$$

Since the R.H.S of (**) is $12.e^{2t}$, let us try $y_p = a.e^{2t}$. Then $\dot{y}_p = 2a.e^{2t}$ & $\ddot{y}_p = 4a.e^{2t}$.

So (**) becomes

$$\ddot{y}_p + \dot{y}_p - 2y_p = 12e^{2t}$$

$$\therefore \quad [4a + 2a - 2a].e^{2t} = 12e^{2t}$$

$$\therefore \quad 4a.e^{2t} = 12e^{2t} \Rightarrow a = 3. \qquad \therefore \; y_p = 3e^{2t} = 3x^2$$

Hence general solution will be

$$y = y_c + y_p = C_1 x + C_2 x^{-2} + 3x^2.$$

Theorem 3 (Cauchy-Euler Equation Theorem)

The nature of the solutions of the ODE

$$a x^2. y'' + bx.y' + c.y = 0 \qquad (*)$$

depends on the roots r_1 & r_2 of the auxiliary equation $a.r(r-1) + b.r + c = 0$. $\qquad (**)$

(a) If r_1 & r_2 are distinct real roots of (**), then

$$y = C_1 . x^{r_1} + C_2 . x^{r_2}, \qquad \text{for } x > 0$$

(b) If $r_1 = r_2$ then r_1 & r_2 will be real and

$$y = [C_1 + C_2 . \ln(x)]. x^{r_1}.$$

(c) If $r_1 = \alpha + i\beta$ is complex, then $r_2 = \alpha - i\beta$ and

$$y = x^{\alpha}.[C_1 . \cos(\beta \ln x) + C_2 . \sin(\beta . \ln x)].$$

Proof: If we put $x = e^t$, then $xD = \Delta$ & $x^2 D = \Delta(\Delta - 1)$

So (*) becomes $[a\Delta(\Delta-1) + b\Delta + c]y = 0$. Hence the aux. eq. will be $a.\Delta(\Delta-1) + b\Delta + c = 0$ which is the same as $a.r(r-1) + b.r + c = 0$.

(a) If r_1 & r_2 are real & distinct then $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ and so $y = C_1 x^{r_1} + C_2 x^{r_2}$

(b) If $r_1 = r_2$ then $y = (C_1 + C_2 . t)e^{r_1 t} = [C_1 + C_2 . \ln(x)]. x^{r_1}$.

(c) And if $r_1 = \alpha + i\beta$, then $y = e^{\alpha t}[C_1 . \cos(\beta t) + C_2 . \sin(\beta t)]$

$$= x^{\alpha}. [C_1 \cos\{\beta . \ln(x)\} + C_2 . \sin\{\beta . \ln(x)\}].$$

§3. Frobenius-series solutions at regular singular points

In this last section we will see that a linear ODE of order 2 always has two linearly independent solutions that are Frobenius-series.

<u>Def</u> A <u>Frobenius-series</u> is any series that can be put in the form $x^r.(a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots)=x^r.\sum_{k=0}^{\infty}a_kx^k$ where r is a constant and $a_0\neq 0$. The a_i's & r are usually real number but we also allow r only to be a complex number.

<u>Ex.1</u> Find two linearly indep. Frobenius-series solution of the ODE $\quad 2x^2.y''-x.y'+(1+x).y=0 \qquad (*)$.

<u>Sol.</u> Let $y(x)=x^r.\sum_{k=0}^{\infty}a_k.x^k=a_0x^r+a_1x^{r+1}+a_2x^{r+2}+\dots+a_nx^{r+n}+\dots$

Then $y'(x)=r.a_0x^{r-1}+(r+1)a_1.x^r+\dots+(r+n+1)a_{n+1}.x^{r+n}+\dots$

& $y''(x)=r(r-1).a_0x^{r-2}+(r+1)ra_1x^{r-1}+\dots+(r+n+1)(r+n+2).a_{n+2}.x^{r+n}+\dots$

So $(*)$ becomes

$$\begin{aligned}
&2r(r-1).a_0.x^r+2(r+1)(r).a_1x^{r+1}+\dots+2(r+n-1)(r+n)\,a_n.x^{r+n}+\dots\\
&-r.a_0.x^r-(r+1).a_1.x^{r+1}-\dots-(r+n).a_n.x^{r+n}+\dots\\
&-a_0.x^r+a_1.x^{r+1}+\dots+a_n.x^{r+n}+\dots\\
&+a_0.x^{r+1}+\dots+a_{n-1}.x^{r+n}+\dots\equiv 0.
\end{aligned}$$

$$\therefore\ [2r(r-1)-r+1]a_0.x^r+\{[2(r+1)(r)-(r+1)+1]a_1+a_0\}x^{r+1}$$
$$+\dots+\{[2(r+n-1)(r+n)-(r+n)+1].a_n+a_{n-1}\}x^{r+n}+\dots\equiv 0$$

Since $a_0\neq 0$, $2r(r-1)-r+1=0$. This is called the <u>indicial</u> equation of the ODE $(*)$ and it gives the possible values of r for which a Frobenius-series solution exist.

So $\quad 2r^2-2r-r+1=0 \Rightarrow 2r^2-3r+1=0$

$\therefore\ (2r-1)(r-1)=0 \Rightarrow r=1$ or $1/2$. Put $r_1=1$ & $r_2=1/2$.

Then there will be two possible Frobenius-series solutions, namely

$$x^1 . (a_0 + a_1 x + a_2 x^2 + \cdots + a_n . x^n + \cdots)$$

& $$x^{1/2} (b_0 + b_1 x + b_2 x^2 + \cdots + b_n . x^n + \cdots) .$$

Here we have written the coefficients as a_n and b_n but we should have really written them as $a_n(r_1)$ & $a_n(r_2)$ – but that might lead to some confusion.

Now let us find a_n & b_n. We know that a_0 & b_0 will be non-zero and arbitrary. We also know that

$$[2(r+n-1)(r+n) - (r+n) + 1]\, a_n(r) + a_{n-1}(r) = 0$$

for all $n \geq 1$. If we put $r = r_1 = 1$, we will $a_n = a_n(r_1)$ and if we put $r = r_2 = 1/2$, we will get $b_n = a_n(r_2)$.

Now $$[\{2(r+n-1) - 1\}(r+n) + 1]\, a_n(r) = -a_{n-1}(r)$$

$$\therefore [2(r+n)(r+n) - 3(r+n) + 1]\, a_n(r) = -a_{n-1}(r)$$

$$\therefore [2(r+n)^2 - 3(r+n) + 1] . a_n(r) = -a_{n-1}(r)$$

$$a_n(r) = \frac{-a_{n-1}(r)}{2(r+n)^2 - 3(r+n) + 1}$$

Putting $r = r_1 = 1$, we get

$$a_n = a_n(r_1) = \frac{-a_{n-1}}{2(1+n)^2 - 3(1+n) + 1} = \frac{a_{n-1}}{(1+n)(2+2n-3)+1}$$

$$= \frac{-a_{n-1}}{2n^2 + 4n - 3n + 2 + 1 - 3} = \frac{a_{n-1}}{2n^2 + n} = \frac{a_{n-1}}{n(2n+1)}$$

$$\therefore a_1 = \frac{-a_0}{1(2+1)} = -\frac{a_0}{3} = \frac{(-1)^1 a_0}{1!\,(3)}$$

$$a_2 = \frac{-a_1}{2(4+1)} = \frac{-a_1}{10} = \frac{a_0}{30} = \frac{(-1)^2 a_0}{2!\,(3)(5)}$$

$$a_3 = \frac{-a_2}{3(6+1)} = \frac{-a_0}{3(7)(30)} = \frac{(-1)^3 . a_0}{3!\,(3)(5)(7)}$$

$$a_4 = \frac{-a_3}{4(8+1)} = \frac{a_0}{4(9)(3)(7)(30)} = \frac{(-1)^4 . a_0}{4!\,(3)(5)(7)(9)} .$$

From this it is not difficult to see that in general

$$a_n = \frac{(-1)^n a_0}{n!\,(3)(5)(7)\cdots(2n+1)} = \frac{(-1)^n.(n!).2^n a_0}{n!\,(2n+1)!} = \frac{(-1)^n 2^n a_0}{(2n+1)!}$$

$$\therefore\ y_1(x) = x^1.\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots\right)$$

$$= a_0 . x^1.\left(\frac{1}{1!} - \frac{2x}{3!} + \frac{2^2x^2}{5!} - \cdots + \frac{(-1)^n.2^n}{(2n+1)!}\right.$$

$$= a_0 x^{1/2}\left(\frac{x^{1/2}}{1!} - \frac{2^1.x^{3/2}}{3!} + \frac{2^2.x^{3/2}}{5!} - \cdots + \frac{(-1)^n.2^n}{(2n+1)!} x^{(2n+1)/2} + \cdots\right)$$

Now if we put $r = r_2 = 1/2$, we get

$$b_n = a_n(r_2) = \frac{-b_{n-1}}{2(1/2+n)^2 - 3(1/2+n)+1} = \frac{-b_{n-1}}{2n^2+2n+\frac{1}{2}-3n-\frac{3}{2}-1}$$

$$= \frac{-b_{n-1}}{2n^2-n} = \frac{-b_{n-1}}{n(2n-1)}.$$

$$\therefore\ b_1 = \frac{-b_0}{1(2-1)} = \frac{-b_0}{1} = \frac{(-1)^1.b_0}{1!}$$

$$b_2 = \frac{-b_1}{2(3)} = \frac{b_0}{2(3)} = \frac{(-1)^2 b_0}{2!(3)}$$

$$b_3 = \frac{-b_2}{3(5)} = \frac{-b_0}{2!(3)(3)(5)} = \frac{(-1)^3 b_0}{3!(3)(5)}$$

$$b_4 = \frac{-b_3}{4(7)} = \frac{b_0}{3!(4)(3)(5)(7)} = \frac{(-1)^4 b_0}{4!(3)(5)(7)}.$$

So
$$b_n = \frac{(-1)^n b_0}{n!(3)(5)\cdots(2n-1)} = \frac{(-1)^n.n!\,2^n b_0}{n!\,(2n-1)!(2n)} = \frac{(-1)^n.2^n b_0}{(2n)!}$$

$$\therefore\ y_2(x) = x^{1/2}\left(b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots\right)$$

$$= b_0.x^{1/2}.\left(1 - \frac{2.x}{2!} + \frac{2^2x^2}{4!} - \cdots + \frac{(-1)^n.2^n}{(2n)!} + \cdots\right)$$

It is not difficult to see that

$$y_1(x) = a_0.\sqrt{x}.\sin(\sqrt{2x}) \quad \& \quad y_2(x) = b_0.\sqrt{x}.\cos(\sqrt{2x}).$$

So two linearly independent solution of (*) will be $\sqrt{x}.\sin(\sqrt{2x})$ and $\sqrt{x}.\cos(\sqrt{2x})$ for $x > 0$.

We were lucky that these solutions also are valid for $x = 0$.

Now we were very lucky that everything worked out nicely and we got the complete solution of (*), but in general we will not be so fortunate. Also the process is quite complicated – so from now on, we shall concentrate on find the <u>indicial equation</u> and find the <u>form</u> of two linearly independent solutions to a given linear ODE about a regular singular point. Our first result will give us an easy way to find the indicial equation by using the corresponding Cauchy-Euler ODE that is associated with the given linear ODE.

Let $x_0 = 0$ be a regular singular point of the ODE

$$x^2 y'' + x . P(x) . y' + Q(x) . y = 0 \quad \cdots (*)$$

Here $P(x)$ & $Q(x)$ must be analytic functions at $x_0 = 0$ because $x_0 = 0$ is a regular singular point of (*). So

$P(x) = p_0 + p_1 . x + p_2 . x^2 + \cdots + p_n . x^n + \cdots$ and

$Q(x) = q_0 + q_1 . x + q_2 . x^2 + \cdots + q_n . x^n + \cdots$

<u>Def</u> The Cauchy-Euler ODE that is associated with (*) is

$$x^2 . y'' + p_0 . x . y' + q_0 . y = 0 \quad \cdots (**)$$

Note that it is possible for p_0 or q_0 to be zero.

Now if we put $x = e^t$ and $D = \frac{d}{dx}$ & $\Delta = \frac{d}{dt}$, then $xD = \Delta$ and $x^2 D^2 = \Delta(\Delta - 1)$. So (**) becomes

$$[x^2 D^2 + p_0 . xD + q_0]\, y = 0$$

So $$[\Delta(\Delta-1) + p_0 . \Delta + q_0]\, y = 0$$

$\therefore$ $\Delta(\Delta - 1) + p_0 . \Delta + q_0 = 0$ will be the auxiliary equation for (**). This same auxiliary equation will be the indicial equation of (*).

Theorem 4 (Indicial equation theorem)

The indicial equation of the linear ODE

$$x^2 . y'' + x . P(x) . y' + Q(x) . y = 0 \quad \cdots (*)$$

is the same as the auxiliary equation of the associated Cauchy-Euler ODE

$$x^2 . y'' + x . p_0 . y' + q_0 . y = 0 \quad \cdots (**)$$

In other words, the indicial equation of $(*)$ will be

$$r(r-1) + p_0 . r + q_0 = 0 .$$

Here p_0 = constant term in the expansion of $P(x)$ and q_0 = constant term in the expansion of $Q(x)$.

Ex.1 Solve the indicial equation of the linear ODE

$$x^2 . y'' + x . (2 + 3x^2) y' + (-2 - x^3) . y = 0 \quad \cdots (*)$$

about $x_0 = 0$.

Sol. The associated Cauchy-Euler ODE is

$$x^2 . y'' + x . \cdot . y' - 2 . y = 0 \quad \cdots (**)$$

So the auxiliary equation of $(**)$ is

$$\Delta(\Delta - 1) + 2\Delta - 2 = 0 .$$

Hence the indicial equation of $(*)$ will be

$$r(r-1) + 2r - 2 = 0 . \quad \text{So } r^2 + r - 2 = 0$$

$\therefore (r-1)(r+2) = 0 \Rightarrow r = 1$ or -2. So $r_1 = 1$ & $r_2 = -2$

Ex.2 Solve the indicial equation of the ODE

$$2x^2 . y'' + x . (x + 3x^2) . y' - (4 + x^3) y = 0 \quad \cdots (*)$$

Sol. Assoc. Cauchy-Euler ODE is $2x^2 y'' + x(0) . y' - 4 . y = 0$

$\therefore$ Aux. eq. is $2\Delta(\Delta - 1) + 0 . \Delta - 4 = 0$

$\therefore$ Indicial eq. is $2r(r-1) + 0 . r - 4 = 0$

$\therefore 2r^2 - 2 - 4 = 0 . \quad \therefore r^2 - 1 - 2 = 0$

$\therefore (r+1)(r-2) = 0 \quad \therefore r = 2$ or -1.

So $r_1 = 2$ & $r_2 = -1$. Note r_1 is always the bigger root.

Ex.3 Find the indicial equation of the ODE
$$2x^2 \cdot y'' - x \cdot y' + (x-5) \cdot y = 0 \quad \cdots \quad (*)$$
and use it to find the form of 2 lin. indep. Frobenius-series solutions about $x_0 = 0$ of $(*)$.

Sol. The Cauchy-Euler ODE associated with $(*)$ is
$$2x^2 \cdot y'' - 1 \cdot x \cdot y' - 5 \cdot y = 0 \quad \cdots (**)$$
So the auxiliary equation of $(**)$ will be
$$2\Delta(\Delta-1) - \Delta - 5 = 0$$
So the indicial equation of $(*)$ will be
$$2r(r-1) - r - 5 = 0$$
$$\therefore \quad 2r^2 - 3r - 5 = 0 \Rightarrow (2r-5)(r+1) = 0$$
$\therefore$ $r_1 = 5/2$ and $r_2 = -1$. Since $r_1 - r_2 = 7/2 \notin \mathbb{N}$, we know from Theorem 5A that there will be two lin. indep. Frobenius-series solution of the form
$$y_1(x) = x^{5/2} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \quad \& \quad y_2(x) = x^{-1} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$$
with $a_0 = b_0 = 1$. And indeed, it can be shown that
$$y_1(x) = x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \cdots\right)$$
$$y_2(x) = x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \cdots\right)$$
by doing it the hard way as on p. 254-256 in the textbook.

Def. Let p be a real constant. The Bessel ODE of order p is the ODE $x^2 \cdot y'' + x \cdot y' + (x^2 - p^2) y = 0$.

Ex.4 Find the form of two lin. indep. solutions of the Bessel ODE $x^2 \cdot y'' + x \cdot y' + x^2 \cdot y = 0 \cdots (*)$ of order 0.

Here $x_0 = 0$ is a regular singular point of $(*)$.

Theorem 5A (Frobenius theorem with real roots)

Suppose the roots of the indicial equation of the linear ODE

$$x^2 \cdot y'' + x \cdot P(x) \cdot y' + Q(x) \cdot y = 0 \quad \cdots \quad (*)$$

are real and $r_1 \geq r_2$, $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ & $\mathbb{N}^+ = \{1, 2, 3, \cdots\}$.

(a) If $r_1 - r_2 \notin \mathbb{N}$, then the ODE (*) has two linearly independent solutions of the form
$y_1(x) = x^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ & $y_2(x) = x^{r_2} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$ with $a_0 = b_0 = 1$ for $x > 0$.

(b) If $r_1 - r_2 = 0$, then (*) has 2 linearly indep. solutions of the form $y_1(x) = x^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ with $a_0 = 1$, and
$y_2(x) = y_1(x) \cdot \ln(x) + x^{r_1} \cdot \sum_{n=1}^{\infty} b_n \cdot x^n$. Here it is possible for all the b_n's to be zero. Note also b_n starts with $n=1$.

(c) If $r_1 - r_2 \in \mathbb{N}^+$, then (*) has two lin. indep. solutions of the form $y_1(x) = x^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ with $a_0 = 1$ and
$y_2(x) = A \cdot y_1(x) \cdot \ln(x) + x^{r_2} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$ with $b_0 = 1$.
Here it is possible for A to be zero.

Theorem 5B (Frobenius theorem with complex roots)

Suppose the roots of the indicial equation of (*) are complex. Then let $r_1 = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$. The other root must be $\alpha - i\beta$. The ODE (*) will have two linearly independent solutions of the form

$$y_1(x) = x^{\alpha} \cdot \cos(\beta \ln x) \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \text{ with } a_0 = 1, \text{ and}$$

$$y_2(x) = x^{\alpha} \cdot \sin(\beta \ln x) \cdot \sum_{n=0}^{\infty} b_n \cdot x^n \text{ with } b_0 = 1.$$

The "cos" & "sin" are here because

$$\begin{aligned} x^{\alpha + i\beta} &= x^{\alpha} \cdot x^{i\beta} = x^{\alpha} \cdot (e^{\ln x})^{i\beta} \\ &= x^{\alpha} \cdot e^{i\beta \ln x} \\ &= x^{\alpha} \cdot \{\cos(\beta \ln x) + i \sin(\beta \ln x)\}. \end{aligned}$$

Sol. The associated Cauchy-Euler ODE is

$$x^2 y'' + 1.x.y' + 0.y = 0 \qquad (**)$$

The aux. equation of (**) is $\Delta(\Delta-1) + \Delta + 0 = 0$

Hence the indicial equation of (*) will be $r(r-1)+r=0$.

So $r^2 - r + r = 0 \Rightarrow r^2 = 0 \Rightarrow r_1 = 0 \ \&\ r_2 = 0$.

Now by Theorem 5A, we see that there will be two linearly indep. solutions of the form

$$y_1(x) = x^0 . \sum_{n=0}^{\infty} a_n . x^n \quad \text{with } a_0 = 1, \text{ and}$$

$$y_2(x) = y_1(x) . \ln(x) + x^0 . \sum_{n=1}^{\infty} b_n . x^n .$$

And, indeed it can be shown that

$$y_1(x) = J_0(x) = x^0 . \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} . \left(\frac{x}{2}\right)^{2n}, \text{ and}$$

$$y_2(x) = J_0(x) . \ln(x) + x^0 . \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2} . \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) . \left(\frac{x}{2}\right)^{2n}$$

Ex. 5 Find the form of 2 lin. indep. Frobenius-series solutions of the ODE $\quad x^2 . y'' - x . y' - (5/4 + x^2) . y = 0 \quad \dots (*)$ about $x_0 = 0$.

Sol. The Cauchy-Euler ODE associated with (*) is

$$x^2 . y'' - 1 . x . y' - (5/4) . y = 0 \qquad \dots (**)$$

The aux. eq. of (**) is $\Delta(\Delta-1) - \Delta - 5/4 = 0$.

So the indicial eq. of (*) is $r(r-1) - r - 5/4 = 0$.

$\therefore \ r^2 - 2r - 5/4 = 0$. So $(r - 5/2)(r + 1/2) = 0$

$\therefore \ r_1 = 5/2 \ \&\ r_2 = -1/2$ because $r_1 \geq r_2$ always.

Since $r_1 - r_2 = 5/2 - (-1/2) = 3 \in \mathbb{N}^+$, there will be two lin. indep. solutions of the form

$$y_1(x) = x^{5/2} . \sum_{n=0}^{\infty} a_n . x^n \quad \text{with } a_0 = 1, \text{ and}$$

$$y_2(x) = A . y_1(x) . \ln(x) + x^{-1/2} . \sum_{n=0}^{\infty} b_n . x^n \quad \text{with } b_0 = 1 \ \&\ A \in \mathbb{R}.$$

And, indeed, it can be shown that

$$y_1(x) = x^{5/2}\left(1 + \frac{x^2}{2(5)} + \frac{x^4}{2(5)(4)(7)} + \frac{x^6}{2(5)(4)(7)(6)(9)} + \cdots\right)$$

& $y_2(x) = 0 \cdot y_1(x) \cdot \ln(x) + x^{-1/2}\left(1 - \frac{x^2}{2} - \frac{x^4}{2(4)} - \frac{x^6}{2(4)(3)(6)} - \cdots\right)$

Here we were just lucky, most of the time $A \neq 0$.

Ex. 6 Find the form of two Frobenius-series solutions of the ODE $x^2 \cdot y'' - x(1+2x) \cdot y' + (5-x) \cdot y = 0 \quad \cdots (*)$ about $x_0 = 0$.

Sol. Here the Cauchy-Euler ODE associated with $(*)$ is

$$x^2 \cdot y'' - 1 \cdot x \cdot y' + 5 \cdot y = 0 \quad \cdots (**).$$

So aux. eq. of $(**)$ is $\Delta(\Delta-1) - \Delta + 5 = 0$. Hence the indicial eq. of $(*)$ will be $r(r-1) - r + 5 = 0$.

$\therefore \quad r^2 - 2r + 5 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\therefore \quad r_1 = 1+2i$ and $r_2 = 1-2i$. So two linearly independent Frobenius-series solutions of $(*)$ will be

$$y_1(x) = x^1 \cdot \cos(2\ln x) \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$y_2(x) = x^1 \cdot \sin(2\ln x) \cdot \sum_{n=0}^{\infty} b_n \cdot x^n \quad \text{with } a_0 = 1 = b_0.$$

Ex. 7 It can be shown that the ODE

$$x^2 \cdot y'' + x \cdot (1-x) \cdot y' - (1+3x) \cdot y = 0$$

has two Frobenius-series solutions as shown below.

$$y_1(x) = x^1 \cdot \sum_{n=0}^{\infty} \frac{(n+3)}{3 \cdot (n!)} \cdot x^n \quad \text{and}$$

$$y_2(x) = -3 \cdot y_1(x) \ln(x) + x^{-1} \cdot \left\{1 - 2x + \sum_{n=2}^{\infty} \frac{1 - (n+1) H_{n-2}}{(n-2)!} x^n\right\}$$

where $H_n = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$.

This shows that the "A" in Theorem 5A (c) can be non-zero.