MAP 2302 - Differential Equations

Test #1 - Spring 2017

Florida International University

Time: 75 min.

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. Begin each of the 6 questions on 6 separate pages.

15) (a) Define what it means for the ODE \( \frac{dy}{dx} = f(x,y) \) to be homogeneous.
(b) Find the general solution of the ODE \( (3y^3 + x^2y) \, dx - 2xy^2 \, dy = 0 \).

15) 2(a) Let \( F(x,y) \) be a function in a domain \( D \) of the plane. Define what is the total differential of \( F \) in \( D \) (when it exists).
(b) Find the general solution of the differential equation
\[
\{y^2 \cdot \sec(x) \tan(x) + 4x\} \, dx + \{3y^2 \cdot \sec(x) - 2 \cdot e^{-2y}\} \, dy = 0.
\]

15) 3(a) Let \( F(x,y,a) = 0 \) be a one-parameter family of differentiable curves in the xy-plane. Define what is an orthogonal trajectory of this family.
(b) Find the solution of the ODE \( \frac{dy}{dx} - 2y = 6 \cdot e^x \) with \( y(0) = 3 \).

15) 4. Find the general solution to the differential equation \( \frac{dy}{dx} - y/x = 3x \cdot y^{-1} \).

20) 5. The population of a colony of micro-organisms satisfy the logistic equation \( \frac{dP}{dt} = 2P(1 - P/4000) \), where \( t \) is measured in hours.
(a) If \( P(0) = 3000 \), find the population after \( t \) hours.
(b) How long will it take for the population to reach 3600 ?

20) 6. A box of mass 4 kg is moving horizontally with velocity 20 \( ms^{-1} \) when it lands on a horizontal surface with coefficient of friction \( \mu = 1/2 \). If the acceleration due to gravity \( g \) is 10 \( ms^{-2} \) and the air resistance is \( \lambda v \) where \( \lambda = 2 \) \( kgs^{-1} \),
(a) For how long will the box slide on the surface?
(b) How far will the box slide on the surface?
1(a) The ODE $dy/dx = f(x,y)$ is homogeneous if we can find a function $g$ such that $f(x,y) = g(y/x)$.

(b) $(3y^3 + x^2y)dx - 2xy^2dy = 0$, so $2xy^2dy = 3y^3 + x^2y$;
\[
\frac{dy}{dx} = \frac{3y^3 + x^2y}{2xy^2} = \frac{3y}{2} \left( \frac{y}{x} \right) + \frac{1}{2} \left( \frac{y}{x} \right)^2.
\]
Put $y = xv$. Then $\frac{dy}{dx} = v + x \frac{dv}{dx}$ and $v = y/x$. So
\[
\frac{v + x \frac{dv}{dx}}{2v^2} = \frac{3v}{2} + \frac{1}{2v}, \quad \therefore \quad \frac{xdv}{dx} = \frac{v}{2} + \frac{1}{2v} = \frac{v^2 + 1}{2v}.
\]
\[
2v \frac{dv}{dx} = \frac{dx}{v^2 + 1}. \quad \therefore \quad \frac{v^2 + 1}{x} \frac{dx}{v} = \int 2v \frac{dv}{2v^2 + 1} = \frac{\ln(v^2 + 1)}{2} = \ln(x) + C.
\]
\[
\therefore \quad \frac{v^2 + 1}{x} = e^{\ln(x) + C} = e^C e^{\ln(x)} = A x \quad \text{where} \quad A = e^C.
\]
\[
(y/x)^2 + 1 = Ax. \quad \therefore \quad y^2 = x^2(Ax - 1).
\]

2(a) The total differential of $F$ is defined by $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$.

(b) \[
\frac{\partial}{\partial y} (y^3 \sec \tan x + 4x) = 3y^2 \sec \tan x \quad \text{and} \quad \frac{\partial}{\partial y} (3y^2 \sec x - 2e^{-2y}) = 3y^2 \sec \tan x;
\]
so the ODE is exact. We will find a function $F$ such that $dF = Mdx + Ndy$.
\[
\frac{\partial F}{\partial x} = (y^3 \sec \tan x + 4x) \quad \text{and} \quad \frac{\partial F}{\partial y} = 3y^2 \sec x - 2e^{-2y}.
\]
\[
F = \int (y^3 \sec \tan x + 4x) \, dx = y^3 \sec x + 2x^2 + \phi(y),
\]
\[
\frac{\partial F}{\partial y} = 3y^2 \sec x + 0 + \phi'(y). \quad \therefore \quad \phi'(y) = -2e^{-2y} \quad \therefore \quad \phi(y) = e^{2y} + C_1. \quad \text{Thus}
\]
\[
F = y^3 \sec x + 2x^2 + 2e^{-2y} + C_1. \quad \text{But} \quad dF = Mdx + Ndy = 0.
\]
So $F = C_2$. Hence $y^3 \sec x + 2x^2 + 2e^{-2y} + C_1 = C_2$. Thus
\[
\text{the solution is } y^3 \sec x + 2x^2 + 2e^{-2y} = C \quad \text{where} \quad C = C_2 - C_1.
\]

3(a) An orthogonal trajectory of the family is any curve in the plane which intersects each curve in the family at right angles.
3(b) \[
\frac{dy}{dx} - 2 \cdot y = 6e^x. \text{ This is a linear first order ODE with } p(x) = -2. \text{ So integrating factor is } e^{\int -2 \, dx} = e^{-2x}.
\]
\[
\therefore e^{-2x} \frac{dy}{dx} - 2e^{-2x} \cdot y = 6e^x \cdot e^{-2x} = 6e^{-x}.
\]
\[
\frac{d}{dx} (ye^{-2x}) = 6e^{-x} \quad \therefore \quad ye^{-2x} = -6e^{-x} + C.
\]
\[
y = -6e^x + Ce^{2x}. \text{ But } y(0) = 3, \text{ so } 3 = -6 + C.
\]
\[
C = 9. \text{ Hence } y = 9e^{2x} - 6e^x.
\]

4. \[
\frac{dy}{dx} - \frac{y}{x} = 3x \cdot y^{-1}. \text{ This is a Bernoulli ODE with index } \alpha = -1. \text{ So multiply both sides of the ODE by } (1-\alpha) \cdot y^{-\alpha} \text{ and put } \upsilon = y^{1-\alpha}. \text{ If we do this correctly the } y \text{ will disappear from the RHS of the ODE.}
\]
\[
1-\alpha = 2, \text{ so } 2y \frac{dy}{dx} - 2y, y = 3x, y^{-1} = 2y
\]
\[
\therefore \frac{2y}{dx} - 2 \cdot y^2 = 6x. \text{ Now put } \upsilon = y^2.
\]
Then \[
\frac{d\upsilon}{dx} = 2y \frac{dy}{dx}. \text{ So } \frac{d\upsilon}{dx} - 2 \cdot \upsilon = 6x. \text{ The integrating factor is } e^{\int 2x \, dx} = e^{-2\ln x} = e^{\ln(x^2)}
\]
\[
\therefore \frac{d}{dx} (x^{-2} \upsilon) = \frac{6}{x} \quad \therefore \quad x^{-2} \upsilon = 6 \ln x + C
\]
Hence \[
\upsilon = x^2(6 \ln x + C). \therefore \quad y^2 = x^2(6 \ln x + C).
\]

5. We have \[
\frac{dP}{dt} = 2P \cdot (1 - P/4000) = 2 \cdot P \cdot \frac{4000 - P}{4000 - P}
\]
\[
\therefore \quad \frac{4000 \, dP}{P(4000 - P)} = 2 \, dt.
\]
Let \[
\frac{4000}{P(4000 - P)} = \frac{A}{P} + \frac{B}{4000 - P}
\]
\[
\therefore \quad (\frac{1}{P} + \frac{1}{4000 - P}) \, dP = 2 \, dt.
\]
Then \[
4000 = A(4000 - P) + BP
\]
Putting \( P = 0 \), gives \( A = 1 \)
Putting \( P = 4000 \), gives \( B = 1 \)
\[
\ln P - \ln(4000 - P) = 2t + C.
\]
\[
\therefore \quad \ln \left( \frac{P}{4000 - P} \right) = 2t + C. \text{ But } P(0) = 3000, \text{ so}
\]
\[
\ln \left( \frac{3000}{4000 - 3000} \right) = 2(0) + C \quad \therefore \quad C = \ln(3). \text{ Hence}
\]
\[
\ln \left[ \frac{P}{(4000 - P)} \right] = 2t + \ln(3).
\]
5(a) \[ \frac{P}{(4000 - P)} = e^{2t + \ln 3} = e^{\ln 3 \cdot 2t} = 3e^{2t} \]

\[ (4000 - P)P = e^{-2t/3} \]

\[ 4000 = 1 + e^{-2t} \]

\[ \frac{P}{3} = \frac{4000}{3} = 3 + e^{-2t} \]

\[ 12000 = P(3 + e^{-2t}) \]

\[ P(t) = 12000/(3 + e^{-2t}) \]

(b) If \( P(t) = 3600 \), then \( \frac{P}{4000 - P} = 3e^{-2t} \). So

\[ 3600 = 3e^{2t} \]

\[ \frac{4000 - 3600}{400} = \frac{3600}{400} = 3e^{2t} \text{ and thus} \]

\[ e^{2t} = 9 \]

\[ t = \frac{\ln 3}{2} \text{ hrs} \]

6(a)

\[ N = mg \]

\[ x(t) = 0 \ m \]

\[ v(t) = 20 \ m/s \]

\[ v(t) = 0 \]

From Newton's 2nd law, we have \( \frac{dv}{dt} = -\mu mg - A v \).

So \( 4 \cdot \frac{dv}{dt} = -\frac{1}{2} \cdot 4.10 - 2.10 \cdot 2 \cdot v \).

\[ 4 \cdot \frac{dv}{dt} = -\frac{1}{2} (10 + v) \]

\[ \frac{dv}{dt} = -\frac{1}{2} dt \]

\[ \ln (v + 10) = -\frac{t}{2} + C \]

\[ v + 10 = e^{-\frac{t}{2} + C} \]

\[ v + 10 = e^{-\frac{t}{2}} \cdot e^C \]

\[ A = e^C \]

But \( v(0) = 20 \), so \( 20 + 10 = A \cdot e^{-\frac{0}{2}} \).

\[ A = 30 \]

\[ v + 10 = 30e^{-\frac{t}{2}} \]. The box will slide until \( v(t) = 0 \). So this will be until \( 0 + 10 = 30e^{-\frac{t}{2}} \).

\[ 10 = 30e^{-\frac{t}{2}} \]

\[ e^{\frac{t}{2}} = 30/10 = 3 \]

\[ \frac{t}{2} = \ln 3 \]

Hence \( t = 2 \ln 3 \) seconds.

(b) \[ v(t) = \frac{dx}{dt} = 30e^{-\frac{t}{2}} - 10 \]. So

\[ x(t) = 30 \cdot (-2) \cdot e^{-\frac{t}{2}} - 10t + C_2 \]

But \( x(0) = 0 \)

So \( 0 = -60 \cdot e^{0} - 10(0) + C_2 \).

\[ C_2 = 60 \]

\[ x(t) = 60 - 60e^{-\frac{t}{2}} - 10t \]

Hence the box will slide a total distance of

\[ x(2\ln 3) = 60 - 60e^{-\frac{(2\ln 3)^2}{2}} - 10(2\ln 3) \]

\[ = 60 - 60e^{\ln(3^{-1})} - 20 \cdot 3 = 60 - (60/3) - 20\ln 3 \]

\[ = 40 - 20 \ln 3 = 20 (2 - \ln 3) \text{ meters.} \]