MAP 2302 - Differential Equations *Florida International University*

TEST #1 - Spring 2017 *TIME: 75 min.*

Answer all 6 questions. ***No calculators, formula sheets, or cell phones are allowed.*** *An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible.* ***Begin each of the 6 questions on 6 separate pages.***

(15)1(a) Define what it means for the *ODE* dy/dx = f(x,y) to be *homogeneous*.

(b) Find the general solution of the *ODE* $(3y^3 + x^2y).dx \; - \; 2xy^2 .dy = 0$.

(15) 2(a) Let F(x,y) be a function in a domain D of the plane. Define what is the *total differential* of F in D (when it exists).

(b) Find the general solution of the differential equation

$$\{y^3.sec(x)tan(x) + 4x\}.dx \quad + \quad \{3y^2.sec(x) - 2.e^{-2y}\}.dy \quad = \quad 0.$$

(15) 3(a) Let F(x,y,a) = 0 be a one-parameter family of differentiable curves in the xy-plane. Define what is *an orthogonal trajectory* of this family.

(b) Find the solution of the ODE $dy/dx - 2.y = 6.e^x$ with $y(0) = 3$.

(15) 4. Find the general solution to the differential equation $dy/dx - y/x = 3x.y^{-1}$.

(20) 5. The population of a colony of micro-organisms satisfy the logistic equation $dP/dt \;=\; 2\,P.(1 - P/4000)$, where t is measured in hours.

(a) If $P(0) = 3000$, find the *population* after t hours.

(b) How *long* will it take for the population to reach *3600* ?

(20) 6. A box of mass 4 *kg* is moving horizontally with velocity $20\ ms^{-1}$ when it lands on a horizontal surface with coefficient of friction $\mu = 1/2$. If the acceleration due to gravity g is $10\ ms^{-2}$ and the air resistance is λv where $\lambda = 2\ kgs^{-1}$,

(a) For how long will the box slide on the surface?

(b) How far will the box slide on the surface?

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Solutions to Test #1 Spring 2017.

1(a) The ODE $dy/dx = f(x,y)$ is homogeneous if we can find a function g such that $f(x,y) = g(y/x)$.

(b) $(3y^3+x^2y)dx - 2xy^2dy = 0$, so $2xy^2\frac{dy}{dx} = 3y^3+x^2y$

$\therefore \frac{dy}{dx} = \frac{3y^3+x^2y}{2xy^2} = \frac{3}{2}\left(\frac{y}{x}\right) + \frac{1}{2}\left(\frac{x}{y}\right) = 3\left(\frac{y}{x}\right) + \frac{1/2}{\left(\frac{y}{x}\right)}$

~~Put $y = xv$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $v = y/x$. So~~

$v + x\frac{dv}{dx} = \frac{3v}{2} + \frac{1}{2v}$. $\therefore x\frac{dv}{dx} = \frac{v}{2} + \frac{1}{2v} = \frac{v^2+1}{2v}$.

$\therefore \frac{2v\,dv}{v^2+1} = \frac{dx}{x}$. So $\ln(v^2+1) = \ln(x) + C$

$\therefore v^2+1 = e^{\ln x + C} = e^C \cdot e^{\ln x} = A \cdot x$ ~~where $A = e^C$~~.

$\therefore \left(\frac{y}{x}\right)^2 + 1 = Ax$. So $y^2 = x^2(Ax-1)$.

2(a) The total differential of F is defined by $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$.

(b) $\frac{\partial}{\partial y}(y^3\sec x\tan x + 4x) = 3y^2\sec x\tan x$ and

$\frac{\partial}{\partial x}(3y^2\sec x - 2e^{-2y}) = 3y^2\sec x\tan x$; so the ODE is exact.

We will find a function F such that $dF = Mdx + Ndy$.

So $\frac{\partial F}{\partial x} = y^3\sec x\tan x + 4x$ and $\frac{\partial F}{\partial y} = 3y^2\sec x - 2e^{-2y}$.

$\therefore F = \int(y^3\sec x\tan x + 4x)\partial x = y^3\sec x + 2x^2 + \varphi(y)$.

$\therefore \frac{\partial F}{\partial y} = 3y^2\sec x + 0 + \varphi'(y)$. But $\frac{\partial F}{\partial y} = 3y^2\sec x - 2e^{-2y}$.

$\therefore \varphi'(y) = -2e^{-2y}$. $\therefore \varphi(y) = e^{-2y} + C_1$. Thus

$F = y^3\sec x + 2x^2 + e^{-2y} + C_1$. But $dF = Mdx + Ndy = 0$.

So $F = C_2$. Hence $y^3\sec x + 2x^2 + e^{-2y} + C_1 = C_2$. Thus the solution is $y^3\sec x + 2x^2 + e^{-2y} = C$ where $C = C_2 - C_1$.

3(a) An orthogonal trajectory of the family is any curve in the plane which intersects each curve in the family at right angles.

3(b) $dy/dx - 2.y = 6e^x$. This is a linear first order ODE with $p(x) = -2$. So integrating factor $= e^{\int -2dx} = e^{-2x}$.

$\therefore e^{-2x}\frac{dy}{dx} - 2.e^{-2x}.y = 6e^x.e^{-2x} = 6e^{-x}$.

$\therefore \frac{d}{dx}(ye^{-2x}) = 6e^{-x}$. $\therefore y.e^{-2x} = -6e^{-x} + C$.

$\therefore y = -6e^x + Ce^{2x}$. But $y(0) = 3$, so $3 = -6 + C$.

$\therefore C = 9$. Hence $y = 9e^{2x} - 6e^x$.

4. $dy/dx - y/x = 3x.y^{-1}$. This is a Bernoulli ODE with index $\alpha = -1$. So multiply both sides of the ODE by $(1-\alpha)y^{-\alpha}$ and put $v = y^{1-\alpha}$. If we do this correctly the y will disappear from the RHS of the ODE.

$1-\alpha = 2$, so $2y\,dy/dx - 2y.\frac{y}{x} = 3x.y^{-1}.2y$

$\therefore 2y\frac{dy}{dx} - \frac{2}{x}y^2 = 6x$. Now put $v = y^2$.

Then $\frac{dv}{dx} = 2y\frac{dy}{dx}$. So $\frac{dv}{dx} - \frac{2}{x}v = 6x$. The integrating factor is $e^{\int(-2/x)dx} = e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2}$.

$\therefore x^{-2}\frac{dv}{dx} - 2x^{-2}.x^{-1}.v = 6x.x^{-2}$

$\therefore x^{-2}\frac{dv}{dx} - 2x^{-3}v = \frac{6}{x}$.

$\therefore \frac{d}{dx}(x^{-2}v) = \frac{6}{x}$. $\therefore x^{-2}v = 6\ln x + C$

Henze $v = x^2(6\ln x + c)$. $\therefore y^2 = x^2(6\ln x + c)$.

5. We have $dP/dt = 2P.(1 - P/4000) = \frac{2.P.(4000-P)}{4000}$

$\therefore \frac{4000\,dP}{P(4000-P)} = 2dt$.

Let $\frac{4000}{P(4000-P)} = \frac{A}{P} + \frac{B}{4000-P}$

Then $4000 = A(4000-P) + BP$

Putting $P=0$, gives $A = 1$

Putting $P = 4000$, gives $B = 1$

$\therefore \left(\frac{1}{P} + \frac{1}{4000-P}\right)dP = 2dt$

$\therefore \ln P - \ln(4000-P) = 2t + C$.

$\therefore \ln\left(\frac{P}{4000-P}\right) = 2t + C$. But $P(0) = 3000$, so

$\ln\left(\frac{3000}{4000-3000}\right) = 2(0) + C$. $\therefore C = \ln(3)$. Hence

$\ln[P/(4000-P)] = 2t + \ln(3)$.

5(a) $\therefore P/(4000-P) = e^{2t+\ln 3} = e^{\ln 3}\cdot e^{2t} = 3e^{2t}$.

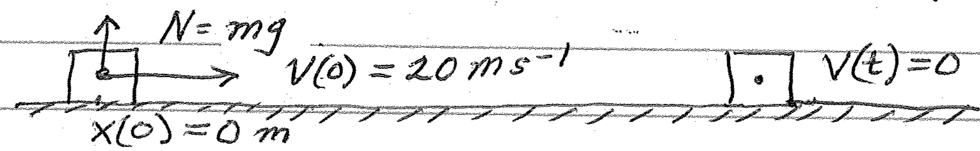
$\therefore (4000-P)/P = e^{-2t}/3$. $\therefore \frac{4000}{P} - 1 = \frac{e^{-2t}}{3}$

$\therefore \frac{4000}{P} = 1 + \frac{e^{-2t}}{3}$. $\therefore \frac{4000}{P} = \frac{3+e^{-2t}}{3}$

$\therefore 12000 = P(3+e^{-2t})$. $\therefore P(t) = 12000/(3+e^{-2t})$.

(b) If $P(t) = 3600$, then $\frac{P}{4000-P} = 3e^{2t}$. So $\frac{3600}{4000-3600} = 3e^{2t}$. So $\frac{3600}{400} = 3e^{2t}$ and thus $9 = 3e^{2t}$. So $e^{2t} = 3$. $\therefore 2t = \ln 3$, so $t = \frac{\ln 3}{2}$ hrs

6(a)



From Newton's 2nd law, we have $\frac{dv}{dt} = -\mu mg - \lambda v$.

So $4\cdot\frac{dv}{dt} = -\frac{1}{2}\cdot 4\cdot 10 - 2\cdot v$. $\therefore \frac{dv}{dt} = -\frac{1}{2}(10+v)$

$\therefore \frac{dv}{v+10} = -\frac{1}{2}dt$. So $\ln(v+10) = -\frac{t}{2} + C_1$

$\therefore v+10 = e^{-t/2+C_1} = e^{C_1}\cdot e^{-t/2} = A\cdot e^{-t/2}$ where $A = e^{C_1}$

But $v(0) = 20$, so $20+10 = A\cdot e^{-0/2}$. $\therefore A = 30$.

$\therefore v+10 = 30e^{-t/2}$. The box will slide until $v(t) = 0$. So this will be until $0+10 = 30e^{-t/2}$

$\therefore 10 = 30e^{-t/2}$. $\therefore e^{t/2} = 30/10 = 3$.

$\therefore t/2 = \ln 3$. Hence $t = 2\ln 3$ seconds

(b) $v(t) = \frac{dx}{dt} = 30e^{-t/2} - 10$. So

$x(t) = 30\cdot(-2)\cdot e^{-t/2} - 10t + C_2$. But $x(0) = 0$

So $0 = -60\cdot e^{-0/2} - 10(0) + C_2$. $\therefore C_2 = 60$.

$\therefore x(t) = 60 - 60e^{-t/2} - 10t$. Hence the box will slide a total distance of

$$x(2\ln 3) = 60 - 60e^{-(2\ln 3)/2} - 10(2\ln 3)$$
$$= 60 - 60\cdot e^{\ln(3^{-1})} - 20\ln 3 = 60 - (60/3) - 20\ln 3$$
$$= 40 - 20\ln 3 = 20(2 - \ln 3) \text{ meters.}$$