MAP 2302 - Differential Equations — *Florida International University*
TEST #2 - Spring 2017 — *TIME: 75 min.*

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(20) 1. Find the solution of each of the following homogeneous differential equations.
(a) $y'' - y' - 2y = 0$ *with* $y(0) = 3$ *and* $y'(0) = 0$.
(b) $y'' + 2y' + y = 0$ *with* $y(0) = 2$ *and* $y'(0) = 1$.

(15) 2(a) Define what is the *Wronskian* of the twice-differentiable functions f_1, f_2, f_3.
(b) Find the *general solution* of the differential equation $y'' - 3y' + 2y = e^x$.

(15) 3. Find the *general solution* of the linear ODE $y'' - 2y' + 2y = 5.\cos(x)$.

(20) 4. Find y_c and give *the minimal form* of the y_p that one should try, for each of the following linear non-homogeneous differential equations.
(a) $(D^2 - 2D + 5)^2\, y = 3x\,.\,e^x\,.\,\cos(2x)$.
(b) $D^2\,(D^2 - 1)(D^2 + 4)y = 2\,.\sin^2(x)$.

(15) 5. Find *a particular solution*, y_p, of the *ODE* $y'' - 2y' + y = (e^x)/(x)$ by using the method of *variation of parameters*.

(15) 6. A body of mass *2 kg* is attached to a *Hooke-type* spring on a horizontal frictionless surface. The natural length of the spring is *5m*, the spring constant *k* is *10 Nm⁻¹*, and the air resistance is λv where λ = *4Nsm⁻¹*. If the spring is stretched by an amount of *3m*, and then set loose from rest at time t=0, find the *amount, x(t), it will be extended* at all subsequent times.

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Solutions to Test #1 Spring 2017

1(a) $y'' - y' - 2y = 0$, so $(D^2 - D - 2)y = 0$. $\therefore$ $(D+1)(D-2)y = 0$

$\therefore$ $y = Ae^{-x} + Be^{2x}$. $y(0) = 3 \Rightarrow A + B = 3 \Rightarrow \cancel{-}A = B-3$

$\therefore$ $y' = -Ae^{-x} + 2Be^{2x}$. $y'(0) = 0 \Rightarrow -A + 2B = 0 \Rightarrow (B-3) + 2B = 0$

$\therefore$ $3B - 3 = 0 \Rightarrow B = 1$. $\therefore$ $A = 3 - B = 2$. $\therefore$ $y = 2e^{-x} + e^{2x}$.

(b) $y'' + 2y' + y = 0$, so $(D^2 + 2D + 1)y = 0$. $\therefore$ $(D+1)^2 y = 0$. Hence

$y = (A + Bx)e^{-x}$. $y(0) = 2 \Rightarrow (A+0)e^0 = 2 \Rightarrow A = 2$.

$y' = B.e^{-x} - (A+Bx)e^{-x}$. $y'(0) = 1 \Rightarrow B - (2+0) = 1 \Rightarrow B = 3$.

$\therefore$ $y = (2+3x)e^{-x}$.

2(a) The Wronskian is defined by $W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$.

(b) Hom. ODE is $y'' - 3y' + 2y = 0$. So $(D^2 - 3D + 2)y = 0$, thus

$(D-1)(D-2)y = 0$. Hence $y_c = C_1 e^x + C_2 e^{2x}$.

Try $y_p = Ax^1.e^x$ (because 1 is a root of the aux. eq. of multiplicity one). Then $y_p' = A.e^x + Axe^x = (A + Ax)e^x$.

$\therefore$ $y_p'' = A.e^x + (A+Ax).e^x = (2A + Ax)e^x$. So $y'' - 3y' + 2y = e^x$

becomes $(2A + Ax).e^x - 3(A + Ax)e^x + 2.Axe^x = e^x$

$\therefore$ $(2A - 3A).e^x + (Ax - 3Ax + 2Ax)e^x = e^x \Rightarrow -A.e^x = 1.e^x \Rightarrow A = -1$

$\therefore$ $y_p = -x.e^x$. Hence $y = y_c + y_p = C_1 e^x + C_2 e^{2x} - x.e^x$.

3. $y'' - 2y' + 2y = 0$ is the Hom. ODE. $\therefore$ $(D^2 - 2D + 2)y = 0$.

$\therefore$ $D = [2 \pm \sqrt{2^2 - 4(1)(2)}]/2 = (2 \pm \sqrt{-4})/2 = (2 \pm 2i)/2 = 1 \pm i$.

$\therefore$ $y_c = C_1 e^x \cos x + C_2 e^x. \sin x$. Try $y_p = A\cos x + B\sin x$

(bec. $\pm i$ are not roots of the auxiliary equation). Then

$y_p' = -A\sin x + B\cos x$ & $y_p'' = -A\cos x - B\sin x$. So

$y'' - 2y' + 2y = 5\cos x$ becomes $(-A\cos x - B\sin x) - 2(-A\sin x + B\cos x)$

$+ 2(A\cos x + B\cos x)$. $\therefore$ $A - 2B = 5$ & $B + 2A = 0$. Hence $A = 1$

and $B = -2$. Thus $y = y_c + y_p = (C_1 \cos x + C_2 \sin x)e^x + (\cos x - 2\sin x)$.

4(a) $(D^2-2D+5)^2 y = 0 \Rightarrow D = \left[2 \pm \sqrt{4-4(5)}\right]/2 = 1 \pm 2i$ (twice).

So $y_c = (C_1 + C_2 x)e^x \cos(2x) + (C_3 + C_4 x)e^x . \sin(2x)$. Since $3x$ is a polynomial of degree 1 and $1 \pm 2i$ are roots of the aux. eq. of mult. 2, the minimal form of y_p that we should try is

$y_p = (A_0 + A_1 x).x^2 . e^x \cos(2x) + (B_0 + B_1 x).x^2 . e^x . \sin(2x)$.

(b) $D^2(D^2-1)(D^2+4)y = 2\sin^2 x = 2(1-\cos(2x))/2 = 1 - \cos(2x)$.

$(D-0)^2(D-1)(D+1)(D-2i)(D+2i) = 0$ is the Homogeneous ODE.

$\therefore y_c = (C_1 + C_2 x)e^{0x} + C_3 e^x + C_4 e^{-x} + C_5 \cos(2x) + C_6 \sin(2x)$.

Since 0 is a root of mult. 2 and $\pm 2i$ are roots of mult. 1 of the aux. eq., the minimal form of y_p that we should try is

$y_p = A_0 . x^2 + A_1 . x^1 . \cos(2x) + B_1 . x^1 \sin(2x)$.

5. $y'' - 2y' + y = 0 \Rightarrow (D^2 - 2D + 1)y = 0 \Rightarrow (D-1)^2 y = 0 \Rightarrow D = 1$ (twice)

$\therefore y_c = C_1 e^x + C_2 . x . e^x$. So take $y_1 = e^x$ and $y_2 = x.e^x$. Then $v_1 y_1 + v_2 y_2$ will be a particular solution of $y'' - 2y' + y = e^x/x$

if $v_1' = \begin{vmatrix} 0 & xe^x \\ e^x/x & (x+1)e^x \end{vmatrix} \Big/ \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = -e^{2x}/e^{2x} = -1$,

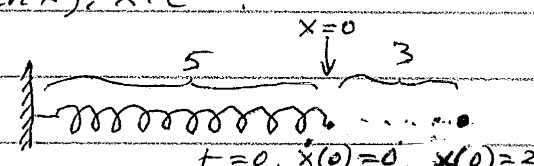
$\Rightarrow v_1 = \int (-1)\,dx = -x$.

& $v_2' = \begin{vmatrix} e^x & 0 \\ e^x & e^x/x \end{vmatrix} \Big/ \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = \frac{1}{x}e^{2x}/e^{2x} = 1/x$.

$\Rightarrow v_2 = \int \frac{1}{x}\,dx = \ln(x)$.

$\therefore y_p = v_1 y_1 + v_2 y_2 = -x . e^x + (\ln x) . x . e^x$.

6. Let $x(t)$ = amount spring is extended at time t. Then the two forces $\lambda \dot{x}$ & kx will work against increasing x. So $m\ddot{x} = -\lambda \dot{x} - kx$.



$\therefore 2\ddot{x} + 4\dot{x} + 10x = 0$. $\therefore (D^2 + 2D + 5)x = 0$. Hence

$D = (-2 \pm \sqrt{4-20})/2 = (-2 \pm 4i)/2 = -1 \pm 2i$. So

$x = A.e^{-t}.\cos(2t) + B.e^{-t}.\sin(2t)$. $x(0) = 3 \Rightarrow A + B.0 = 3$ $\Rightarrow A = 3$

$\dot{x} = -Ae^{-t}.\cos(2t) - B.e^{-t}\sin(2t) - 2Ae^{-t}\sin(2t) + 2Be^{-t}.\cos(2t)$.

$\dot{x}(0) = 0 \Rightarrow -A - B.0 - 2A.0 + 2B = 0 \Rightarrow -3 + 2B = 0$.

$\therefore B = 3/2$. $\therefore x(t) = 3e^{-t}.\cos(2t) + (3/2).e^{-t}.\sin(2t)$.