MAP 2302 - Differential Equations *Florida International University*

TEST 3 - Spring 2017 *TIME: 75 min.*

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(15) 1. Find the general solution of the linear ODE $x^2.y'' + 2x.y' - 2.y = 6.ln(x)$ by first transforming it into a linear constant coefficient ODE in y and t.

(20) 2. For each of the following ODEs, find the indicial equation and the form of two linearly independent Frobenius series solution about $x_0 = 0$.

(a) $x^2.y'' + 3x.y' + (5/4 - 2x).y = 0$.

(b) $x^2.y'' + 2x.y' + (7x - 3/4).y = 0$.

(12) 3. Starting with $\mathscr{L}\{e^{at}\}(s) = 1/(s-a)$, use the properties of the Laplace transform to find (a) $\mathscr{L}\{cos(2t)\}(s)$ (b) $\mathscr{L}\{t.\,cos(2t)\}(s)$

(20) 4. Solve each of the following *IVPs*, by using the Laplace transform.

(a) $y'(t) + y(t) = 6.\,e^t$ with $y(0) = 1$.

(b) $y''(t) + 2.y'(t) + 2.y(t) = 0$ with $y(0) = 1$ & $y'(0) = 2$.

(15) 5. Solve the following system of linear ODEs, by using the Laplace transform.

$x'(t) - 4y(t) = 12$, and

$y'(t) - x(t) = 0$, with $x(0) = 8$ & $y(0) = 0$.

(18) 6. (a) Find the first 5 non-zero terms of the power series solution of the ODE $y'' + 2x.y' - 2y = 0$ with $y(0) = 6$ & $y'(0) = 10$.

(b) Define what it means for 0 to be a *singular point* and what it means for 0 to be a *regular singular point* of the ODE $y'' + P_1(x).y' + P_2(x).y = 0$.

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1. Let $x = e^t$. Then $t = \ln(x)$. If we put $D = \frac{d}{dx}$ & $\Delta = \frac{d}{dt}$, then $xD = \Delta$ and $x^2D^2 = \Delta(\Delta-1)$. So $x^2y'' + 2x.y' - 2y = 6\ln(x)$ becomes $(x^2D^2 + 2xD - 2)y = 6\ln(x)$. $\therefore$ $[\Delta(\Delta-1) + 2\Delta - 2]y = 6t$

Homog. eq. is. $(\Delta^2 + \Delta - 2)y = 0$. $\therefore (\Delta-1)(\Delta+2)y = 0$

$\therefore y_c = C_1e^t + C_2e^{-2t} = C_1x + C_2x^{-2}$.

Try $y_p = A + Bt$. Then $\dot{y}_p = B$ and $\ddot{y}_p = 0$. So $\ddot{y} + \dot{y} - 2y = 6t$ becomes $0 + B - 2(A+Bt) = 6t + 0$. $\therefore -2Bt = 6t$ & $B - 2A = 0$.

$\therefore B = -3$ and $A = B/2 = -3/2$. $\therefore y_p = A + Bt = (-3/2) - 3t = (-3/2) - 3\ln(x)$.

$\therefore y = y_c + y_p = C_1x + C_2x^{-2} - (3/2) - 3\ln(x)$.

2.(a) The ODE is $x^2y'' + 3xy' + (5/4 - 2x)y = 0$. So the associated Cauchy-Euler ODE is $x^2y'' + 3xy' + (5/4)y = 0$. This ODE is equiv. to the equation $[\Delta(\Delta-1) + 3\Delta + 5/4]y = 0$ where $\Delta = \frac{d}{dt}$ and $x = e^t$. So the indicial equation of the given ODE is $r(r-1) + 3r + 5/4 = 0$. $\therefore$ $r^2 + 2r + 5/4 = 0$. So $r = \frac{-2 \pm \sqrt{4 - 4(5/4)}}{2}$ $= -1 \pm (i/2)$. Since the roots are a pair of complex conjugates we will get two linearly independent solutions of the form

$y_1(x) = x^{-1}\cos(\frac{1}{2}\ln x) . \sum_{n=0}^{\infty} a_n . x^n$ with $a_0 = 1$, and

$y_2(x) = x^{-1}\sin(\frac{1}{2}\ln x) . \sum_{n=0}^{\infty} b_n . x^n$ with $b_0 = 1$.

(b) We have $x^2 . y'' + 2x.y' + (7x - 3/4)y = 0$. So the associated Cauchy-Euler ODE is $x^2y'' + 2xy' - (3/4)y = 0$ which is equivalent to $[\Delta(\Delta-1) + 2\Delta - 3/4]y = 0$. So the indicial equation of our original ODE will be $r(r-1) + 2r - 3/4 = 0$. $\therefore r^2 + r - 3/4 = 0$

$\therefore r = [-1 \pm \sqrt{1 + 4(3/4)}]/2 = (-1 \pm 2)/2 = 1/2$ or $-3/2$. Since $r_1 - r_2 = 1/2 - (-3/2) = 2 \in \mathbb{N}^+$, two lin. indep. solutions will be of the form

$y_1(x) = x^{1/2} . \sum_{n=0}^{\infty} a_n . x^n$ with $a_0 = 1$, $b_0 = 1$, and A may or may not be 0.

$y_2(x) = A . y_1(x)\ln(x) + x^{-3/2} . \sum_{n=0}^{\infty} b_n x^n$.

3(a) $\mathcal{L}\{\cos(2t)\}(s) = \mathcal{L}\{(e^{2it}+e^{-2it})/2\} = (1/2)\,[\mathcal{L}\{e^{2it}\} + \mathcal{L}\{e^{-2it}\}]$

$= \frac{1}{2}\left(\frac{1}{s-2i}+\frac{1}{s+2i}\right) = \frac{1}{2}\,\frac{(s+2i)+(s-2i)}{(s-2i)(s+2i)} = \frac{1}{2}\,\frac{2s}{s^2-4i^2} = \frac{s}{s^2+4}$.

(b) $\mathcal{L}\{t\cos(2t)\}(s) = \left(-\frac{d}{dt}\right)\mathcal{L}\{\cos(2t)\} = -\frac{d}{dt}\left(\frac{s}{s^2+4}\right)$

$= -\left[\frac{(s)'(s^2+4) - s.(s^2+4)'}{(s^2+4)^2}\right] = -\frac{[s^2+4-s.(2s)]}{(s^2+4)^2} = \frac{s^2-4}{s^2+4}$.

4(a) $y'(t) + y(t) = 6.e^t$ with $y(0) = 1$. So

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 6\,\mathcal{L}\{e^t\}$. $\therefore$ $s\mathcal{L}\{y\} - y(0) + \mathcal{L}(y) = 6/(s-1)$

$\therefore (s+1).\mathcal{L}\{y\} = y(0) + 6/(s-1) = 1 + 6/(s-1) = (s+5)/(s-1)$.

$\therefore \mathcal{L}\{y\} = \frac{s+5}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$. $\therefore s+5 = A(s+1) + B(s-1)$.

Putting $s=1$, gives us $1+5 = A(1+1)+0 \Rightarrow A = 3$; and putting $s=-1$, gives us $-1+5 = 0 + B(-1-1) \Rightarrow B = -2$. Hence

$\mathcal{L}\{y\} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{3}{s-1} - \frac{2}{s+1}$. $\therefore y(t) = 3e^t - 2e^{-t}$.

(b) $y''(t) + 2y'(t) + 2y(t) = 0$ with $y(0) = 1$ & $y'(0) = 2$.

$\therefore \mathcal{L}\{y''\} + 2\,\mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} = [s^2\mathcal{L}\{y\} - s.y(0) - y'(0)] + 2[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\}$

$= 0$ $\therefore (s^2+2s+2)\mathcal{L}\{y\} = s.y(0) + y'(0) + 2y(0) = s+2+2 = s+4$.

$\therefore \mathcal{L}\{y\} = \frac{s+4}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} + \frac{3}{(s+1)^2+1}$. $y(t) = e^{-t}\cos t + 3e^{-t}\sin t$.

5. $x'(t) - 4.y(t) = 12$, so $[s\mathcal{L}\{x\} - x(0)] - 4.\mathcal{L}\{y\} = \mathcal{L}\{12\} = 12/s$;

$y'(t) - x(t) = 0$, so $[s.\mathcal{L}\{y\} - y(0)] - \mathcal{L}\{x\} = \mathcal{L}\{0\} = 0$.

$\therefore$ $s.\mathcal{L}\{x\} - 4.\mathcal{L}\{y\} = x(0) + 12/s = 8 + (12/s)$

and $s.\mathcal{L}\{y\} - \mathcal{L}\{x\} = y(0) = 0$, because $x(0) = 8$ & $y(0) = 0$.

Now from the last equation we get $\mathcal{L}\{x\} = s.\mathcal{L}\{y\}$.

Substituting in the 2nd to last equation, we get

$s[s.\mathcal{L}\{y\}] - 4.\mathcal{L}\{y\} = 8 + 12/s$. $\therefore (s^2-4)\mathcal{L}\{y\} = \frac{8s+12}{s}$

$\therefore \mathcal{L}\{y\} = \frac{8s+12}{s(s^2-4)} = \frac{8s+12}{s(s-2)(s+2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2}$.

$\therefore 8s+12 = A(s-2)(s+2) + Bs(s+2) + Cs(s-2)$. Putting $s=0$ gives us $12 = A(-2)(2) \Rightarrow A = -3$, putting $s=2$ gives us $28 = B(2)(4)$ $\Rightarrow B = 7/2$, & putting $s=-2$, gives us $-4 = C(-2)(-4) \Rightarrow C = -1/2$.

5. $\therefore \mathcal{L}\{y\} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2} = \frac{-3}{s} + \frac{7}{2}\cdot\frac{1}{s-2} - \frac{1}{2}\cdot\frac{1}{s+2}$. Hence

$y(t) = -3 + (7/2)e^{2t} - (1/2)e^{-2t}$. Check: $y(0) = -3 + \frac{7}{2} - \frac{1}{2} = 0.$ ✓

Also $y'(t) - x(t) = 0$, so $x(t) = y'(t) = 0 + 2(\frac{7}{2})e^{2t} - (-2)(\frac{1}{2})e^{-2t}$.

$\therefore x(t) = 7e^{2t} + e^{-2t}$. Check: $x(0) = 7 + 1 = 8.$ ✓

6. Let $y = \sum_{n=0}^{\infty} a_n . x^n$. Then $y' = \sum_{n=1}^{\infty} n.a_n.x^{n-1} = \sum_{n=0}^{\infty}(n+1).a_{n+1}.x^n$

and $y'' = \sum_{n=2}^{\infty} n(n-1).a_n.x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}.x^n$. Hence

$y'' + 2xy' - 2y = 0$ becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}.x^n + 2x.\sum_{n=1}^{\infty} n.a_n.x^{n-1} - 2.\sum_{n=0}^{\infty} a_n.x^n \equiv 0.$$

$$\therefore [(2)(1)a_2 - 2a_0].x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n-1).a_n].x^n \equiv 0$$

$\therefore 2a_2 - 2a_0 = 0$ and $(n+2)(n+1)a_{n+2} + 2(n-1)a_n = 0$ for $n \geq 1$.

But $y(0) = \sum_{n=0}^{\infty} a_n.0^n = a_0.0^0 = a_0$. So $a_0 = y(0) = 6$. Also

$y'(0) = \sum_{n=0}^{\infty}(n+1)a_{n+1}.0^n = a_1.0^0 = a_1$. So $a_1 = y(0) = 10$. Thus

$2a_2 - 2a_0 = 0 \Rightarrow a_2 = 2a_0/2 = 2(6)/2 = 6$. Also

$(n+2)(n+1)a_{n+2} = -2(n-1)a_n \Rightarrow a_{n+2} = -2(n-1)a_n/(n+2)(n+1)$. Putting

$n=1$ gives us $a_3 = -2(1-1)a_1/(1+2)(1+1) = 0$ because $(1-1)=0$,

$n=2$ gives us $a_4 = -2(2-1)a_2/(2+2)(2+1) = -2(6)/12 = -1$,

$n=3$ gives us $a_5 = -2(3-1)a_3/(3+2)(3+1) = 0$ because $a_3 = 0$,

$n=4$ gives us $a_6 = -2(4-1)a_4/(4+2)(4+1) = -2(3)(-1)/30 = 1/5$.

$\therefore y(x) = 6 + 10x + 6x^2 + 0.x^3 - 1.x^4 + 0.x^5 + (\frac{1}{5})x^6 + \ldots$

(b) $x_0 = 0$ is a singular point of $y'' + P_1(x).y' + P_2(x)y = 0$ if at least one of the two functions $P_1(x)$ & $P_2(x)$ is not analytic at $x_0 = 0$. $x_0 = 0$ is a regular singular point of this ODE if $x_0 = 0$ is a singular point of it and both $(x-0).P_1(x)$ and $(x-0)^2.P_2(x)$ are analytic at $x_0 = 0$.

END