MAP 2302 - Differential Equations — *Florida International University*

TEST #1 - Spring 2018 — *TIME: 75 min.*

Answer all 6 questions. ***No calculators, formula sheets, or cell phones are allowed.*** *An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible.* ***Begin each of the 6 questions on 6 separate pages.***

(15)1 (a) Define what it means for the function f(x,y) to be *homogeneous of degree k.*

(b) Find the general solution of the *ODE* $(xy^2 + 2x^3).dx - 2x^2y.dy = 0.$

(15) 2. Check that $\{y^2.\sin(xy^2) + 6e^{2x}\}.dx + \{2xy.\sin(xy^2) - \sec^2(y)\}.dy = 0$ is an exact ODE and then find its general solution.

(15) 3 (a) Let G(x,y,c) = 0 be a one-parameter family of differentiable curves in the xy-plane. Define what is *an orthogonal trajectory* of this family.

(b) Find the solution of the ODE $dy/dx + (2/x).y = 12.$ with $y(1) = 3.$

(15) 4. Find the general solution to the differential equation $dy/dx - 2y = 4.y^{1/2}$

(20) 5. The population of a colony of micro-organisms satisfy the logistic equation $dP/dt = -P.[(P/3000) - 1]$, where t is measured in hours. If $P(0) = 4{,}000$

(a) find how *long* will it take for the population to reach *3600*, and

(b) find the *population* after *t* hours has elapsed,

(20) 6. A ball of mass 3 *kg* thrown vertically upwards from sea level with velocity $20\ ms^{-1}$. If the acceleration due to gravity, *g* is $10\ ms^{-2}$ and the air resistance is λv where $\lambda = 3\ kgs^{-1}$, find

(a) the time it takes for the ball to reach its greatest height, and

(b) the greatest height above sea-level that the ball reaches.

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Solutions To Test #1   Spring 2018

1 (a) The function $f(x,y)$ is homogeneous of degree k if
$f(\lambda x, \lambda y) = \lambda^k f(x,y)$ for each $\langle x, y\rangle$ in $\mathrm{dom}(f)$.

(b) $(xy^2 + 2x^3)\,dx - (2x^2y)\,dy = 0$, so $2x^2y\,dy = xy^2 + 2x^3$

$\therefore \dfrac{dy}{dx} = \dfrac{xy^2 + 2x^3}{2x^2y} = \dfrac{1}{2}\left(\dfrac{y}{x}\right) + \left(\dfrac{x}{y}\right) = \dfrac{1}{2}\left(\dfrac{y}{x}\right) + 1/\left(\dfrac{y}{x}\right)$.

Put $y = xv$. Then $y/x = v$ and $dy/dx = 1.v + x\,dv/dx$.

So $v + x\dfrac{dv}{dx} = \dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{y}{x}\right) + 1/\left(\dfrac{y}{x}\right) = \dfrac{1}{2}v + \dfrac{1}{v}$.

$\therefore x\dfrac{dv}{dx} = -\dfrac{1}{2}v + \dfrac{1}{v} = -\dfrac{1}{2}(v^2 - 2)/v$. So $\dfrac{2v\,dv}{v^2-2} = -\dfrac{dx}{x}$

$\therefore \ln(v^2 - 2) = -\ln(x) + C = \ln(1/x) + C$

$\therefore v^2 - 2 = e^C . e^{\ln(1/x)} = A.(1/x) \Rightarrow v^2 = 2 + A/x$

$\therefore y^2/x^2 = 2 + A/x$ and so $y^2 = 2x^2 + Ax$.

2 (a) Let $M = y^2 . \sin(xy^2) + 6e^{2x}$ & $N = 2xy.\sin(xy^2) - \sec^2(y)$.

Then $\partial M/\partial y = 2y.\sin(xy^2) + 2xy.y^2.\cos(xy^2) + 0$

$\partial N/\partial x = 2y.\sin(xy^2) + 2xy.y^2.\cos(xy^2) - 0$

$\therefore \partial M/\partial y = \partial N/\partial x$ and so the ODE is exact.

(b) We will find a function $F(x,y)$ such that $dF = M\,dx + N\,dy$

Since $M\,dx + N\,dy = 0$, it will then follow that $dF = 0$, so $F(x,y) = C_1$.

Now $dF = (\partial F/\partial x)dx + (\partial F/\partial y) = M\,dx + N\,dy$. So

$\partial F/\partial x = M = y^2 \sin(xy^2) + 6e^{2x}$ & $\partial F/\partial y = 2xy\sin(xy^2) - \sec^2(y)$

$\therefore F = \int \left(y^2 \sin(xy^2) + 6e^{2x}\right)\partial x = -\dfrac{\cancel{y^2}}{\cancel{y^2}}\cos(xy^2) + 3e^{2x} + \varphi(y)$.

$\therefore \partial F/\partial y = -2xy.[-\sin(xy^2)] + \varphi'(y)$. But $\partial F/\partial y = N$.

So $\varphi'(y) = -\sec^2(y)$. $\therefore \varphi(y) = \int -\sec^2(y)\,dy = -\tan(y) + C_2$.

Hence $F(x,y) = -\cos(xy^2) + 3e^{2x} - \tan(y) + C_2$.

So $3e^{2x} - \cos(xy^2) - \tan(y) + C_2 = C_1$.

$\therefore 3e^{2x} - \cos(xy^2) - \tan(y) = C$ where $C = C_1 - C_2$.

3(a) An orthogonal trajectory of the family $G(x,y,c)=0$ is any curve in the x-y plane which intersects each curve of the given family at right angles.

(b) $dy/dx + (2/x).y = 12$. This is a linear first-order ODE. So integrating factor $= e^{\int (2/x)dx} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2$.

$\therefore\ x^2.(dy/dx) + x^2.(2/x).y = 12x^2$. So $\frac{d}{dx}(y.x^2) = 12x^2$

$\therefore\ y.x^2 = \int 12x^2\,dx = 4x^3 + C$. But $y=3$ when $x=1$.

So $3(1)^2 = 4(1)^3 + C \Rightarrow C=-1$. $\therefore\ y = x^{-2}(4x^3+C) = 4x - 1/x^2$.

4. We have $dy/dx - 2.y = 4y^{1/2}$. This is a Bernoulli ODE with $\alpha = 1/2$. So multiply both sides by $(1-\alpha)y^{-\alpha} = \frac{1}{2}y^{-1/2}$

$\therefore\ (1/2).y^{-1/2}\,dy/dx - 2.(1/2)y^{-1/2}.y = 4.y^{1/2}.(\frac{1}{2}y^{-1/2})$

$\therefore\ (1/2)y^{-1/2}\,dy/dx - y^{1/2} = 2$. Put $v = y^{1-\alpha} = y^{1/2}$

Then $\frac{dv}{dx} = \frac{dv}{dy}.\frac{dy}{dx} = \frac{1}{2}y^{-1/2}.\frac{dy}{dx}$. So we get

$$\frac{dv}{dx} - v = 2 . \qquad \text{Integrating factor} = e^{\int -dx} = e^{-x}$$

$\therefore\ e^{-x}\frac{dv}{dx} - e^{-x}v = 2e^{-x}$. So $\frac{d}{dx}(v.e^{-x}) = 2e^{-x}$

$\therefore\ v.e^{-x} = \int 2.e^{-x}dx = -2e^{-x} + C$

$\therefore\ v = e^{x}(C - 2e^{-x}) = Ce^{x} - 2$

$\therefore\ y^{1/2} = Ce^{x} - 2 \Rightarrow y = (Ce^{x}-2)^2$

5(a) We have $\frac{dP}{dt} = -P\left[(P/3000) - 1\right] = -\frac{P.(P-3000)}{3000}$.

So $\frac{-3000\,dP}{P(P-3000)} = dt$

$$\boxed{\begin{gathered}\text{Put } \frac{-3000}{P(P-3000)} = \frac{A}{P} + \frac{B}{P-3000} \\ \therefore\ -3000 = A(P-3000) + B.P \\ \text{Putting } P=0 \text{, gives us } A=1 \\ \text{Putting } P = 3000\text{, gives us } B=-1\end{gathered}}$$

$\therefore\ \frac{dP}{P} - \frac{dP}{P-3000} = dt$

$\therefore\ \ln P - \ln(P-3000) = t + C$

But $P(0) = 4000$. Hence

$\ln(4000) - \ln(4000-3000) = 0 + C \Rightarrow C = \ln\left(\frac{4000}{1000}\right) = \ln 4$.

$\therefore\ t = \ln(P) - \ln(P-3000) - \ln 4 = \ln\left[\frac{P}{4(P-3000)}\right]$.

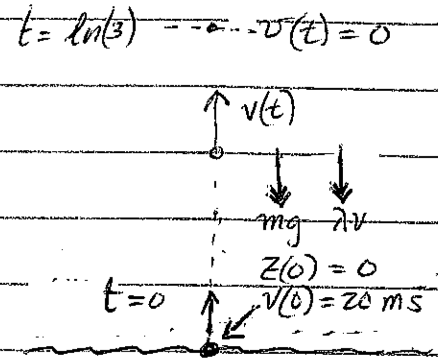
So when $P = 3600$, $t = \ln\left[\frac{3600}{4(3600-3000)}\right] = \ln\left(\frac{3}{2}\right)$ hours.

5(b) We know that $t = \ln \frac{P}{4(P-3000)}$ $\therefore$ $e^t = \frac{P}{4(P-3000)}$

$\therefore$ $4(P-3000)e^t = P \Rightarrow P(4e^t - 1) = 12000e^t$.

$\therefore$ $P(t) = \frac{12000e^t}{4e^t - 1} = \frac{12000}{4 - e^{-t}}$. [Check $P(0) = \frac{12000}{4-1} = 4000$]

6 (a) From Newton's 2nd law, we have



$m\frac{dv}{dt} = -mg - \lambda v$. So

$3\frac{dv}{dt} = -3(10) - 3v$

$\therefore$ $dv = -(10+v)\,dt$

$\therefore$ $\frac{dv}{10+v} = -dt$

$\therefore$ $\ln(10+v) = -t + C_1$. But $v(0) = 20$. So

$\ln(10+20) = -0 + C_1 \Rightarrow C_1 = \ln(30)$.

$\therefore$ $\ln(10+v) = -t + \ln(30) \Rightarrow 10 + v = e^{\ln(30)} \cdot e^{-t} = 30e^{-t}$

$\therefore$ $v(t) = 30e^{-t} - 10$. $v(t) = 0 \Rightarrow 30e^{-t} - 10 = 0$

$\Rightarrow 30e^{-t} = 10 \Rightarrow 3 = e^t$

$\Rightarrow t = \ln 3$.

So ball reaches the greatest height in $\ln(3)$ seconds

(b) $v(t) = \frac{dz}{dt} = 30e^{-t} - 10$. $\therefore$ $z(t) = -30e^{-t} - 10t + C_2$

But $z(0) = 0$. So $0 = -30 - 10(0) + C_2 \Rightarrow C_2 = 30$.

$\therefore$ $z(t) = 30 - 30e^{-t} - 10t$.

$\therefore$ $z(\ln 3) = 30 - 30e^{-\ln 3} - 10\ln 3 = 30 - \cancel{30}^{10} \cdot \frac{1}{\cancel{3}} - 10\ln 3$

$= 20 - 10\ln 3 = 10(2 - \ln 3)$.

So greatest height reached will be $10(2 - \ln 3)$ metres

END