MAP 2302 - Differential Equations | *Florida International University*
TEST #2 - Spring 2018 | *TIME: 75 min.*

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(20) 1. Find the solution of each of the following homogeneous differential equations.
(a) $y'' - 2y' - 8y = 0$ with $y(0) = 5$ and $y'(0) = 2$.
(b) $4y'' - 4y' + y = 0$ with $y(0) = 2$ and $y'(0) = 4$.

(15) 2. (a) Define what it means for the functions $f_1(x), f_2(x), f_3(x)$ to be *linearly independent*.
(b) Find the *general solution* of the differential equation $y'' - y = 8.e^x$.

(15) 3. Find the *general solution* of the linear ODE $y'' + 2y' + 2y = -10.\sin(2x)$.

(20) 4. Find y_c and give *the minimal form* of the y_p that one should try, for each of the following linear non-homogeneous differential equations.
(a) $(D-2)^3.(D^2-4D+5)^2 y = 7.x^2.e^{2x}.\sin(x)$.
(b) $(D-0)^3.(D+2).(D^2+4)^2 y = 4.x.\cos^2(x)$.

(15) 5. Find *a particular solution*, y_p, of the *ODE* $y'' + y = \cot(x)$ by using the method of *variation of parameters*.

(15) 6. A body of mass $2kg$ is attached to a *Hooke-type* spring and suspended from a very high ceiling. The natural length of the spring is $7m$, the spring constant k is $20Nm^{-1}$, and the air resistance is λv where $\lambda = 4Nsm^{-1}$. If the spring is stretched by an amount of $4m$, and then set loose from rest at time t=0, find the *amount, $x(t)$, it will be extended* at all subsequent times. [*Use* $g = 10\ ms^{-2}$]

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Solutions to Test #2 — Spring 2018

1(a) $y'' - 2y' - 8y = 0$, so $(D^2 - 2D - 8)y = 0$. $\therefore$ $(D+2)(D-4)y = 0$.

$\therefore$ $y = Ae^{-2x} + Be^{4x}$. Now $y(0) = 5 \Rightarrow A + B = 5 \Rightarrow -A = B - 5$.

$\therefore$ $y' = -2Ae^{-2x} + 4Be^{4x}$. Also $y'(0) = 2 \Rightarrow -2A + 4B = 2 \Rightarrow 2(B-5) + 4B = 2$

$\therefore$ $6B = 12 \Rightarrow B = 2$. $\therefore$ $A = 5 - B = 3$. $\therefore$ $y = 3e^{-2x} + 2e^{4x}$.

(b) $4y'' - 4y' + y = 0$, so $(4D^2 - 4D + 1)y = 0$. $\therefore$ $(2D-1)^2 = 0 \Rightarrow D = \frac{1}{2}$ (twice)

$\therefore$ $y = A.e^{x/2} + Bxe^{x/2}$. So $y(0) = 2 \Rightarrow A + B.0 = 2 \Rightarrow A = 2$

$\therefore$ $y' = \frac{A}{2}.e^{x/2} + \frac{B}{2}xe^{x/2} + B.1.e^{x/2}$. So $y'(0) = 4 \Rightarrow \frac{A}{2} + 0 + B = 4 \Rightarrow B = 3$

$\therefore$ $y = 2e^{x/2} + 3x.e^{x/2}$.

2(a) The functions $f_1(x), f_2(x), f_3(x)$ are linearly independent if $c_1.f_1(x) + c_2.f_2(x) + c_3.f_3(x) \equiv 0 \Rightarrow c_1 = 0, c_2 = 0$, and $c_3 = 0$.

(b) $y'' - y = 8.e^x$. Homog. ODE is $y'' - y = 0$. $\therefore (D^2 - 1)y = 0$

$\therefore$ $(D-1)(D+1)y = 0 \Rightarrow y_c = C_1.e^{-x} + C_2e^x$. Since $\alpha = 1$ is a root of the aux. eq. of multiplicity 1, try $y_p = a.x^1.e^x$.

Then $y_p' = a.1.e^x + a.x.e^x$ & $y_p'' = a.1.e^x + a.1.e^x + a.x.e^x$ $= 2a.e^x + ax.e^x$. So $y'' - y = 8.e^x$ becomes

$$2a.e^x + ax.e^x - ax.e^x = 8.e^x \Rightarrow 2a.e^x = 8.e^x.$$

$\therefore$ $2a = 8 \Rightarrow a = 4$. $\therefore$ $y_p = 4x.e^x \Rightarrow y = y_c + y_p$

$$= C_1e^{-x} + C_2e^x + 4xe^x.$$

3. $y'' + 2y' + 2y = -10\sin(2x)$. Homog. ODE is $y'' + 2y' + 2y = 0$

$\therefore$ $D^2 + 2D + 2 = 0 \Rightarrow D = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$. $\therefore$ $y_c = C_1e^{-x}\cos x + C_2e^{-x}\sin x$.

Try $y_p = a\cos(2x) + b\sin(2x)$. Then $y_p' = 2b\cos(2x) - 2a\sin(2x)$ and $y_p'' = -4a\cos(2x) - 4b\sin(2x)$. So $y_p'' + 2y_p' + 2y_p = -10\sin(2x)$ becomes $-4a\cos(2x) - 4b\sin(2x) + 4b\cos(2x) - 4a\sin(2x) + 2a\cos(2x)$ $+ 2b\sin(2x) = -10\sin(2x)$. $\therefore$ $(-4a + 4b + 2a)\cos(2x) +$ $(-4b - 4a + 2b)\sin(2x) = -10\sin 2x$. $\therefore$ $4b - 2a = 0$ & $-4a - 2b = -10$.

$\therefore$ $a = 2b \Rightarrow -8b - 2b = -10 \Rightarrow b = 1$. So $a = 2$. Hence

$y_p = a\cos(2x) + b\sin(2x) = 2\cos(2x) + \sin(2x)$. Hence

$y = y_c + y_p = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x + 2\cos(2x) + \sin(2x)$

4(a) $(D-2)^3(D^2-4D+5)^2 y = 0$. So $D = 2$ (thrice) or

$D = [-(-4) \pm \sqrt{16-20}]/2 = (4 \pm 2i)/2 = 2 \pm i$ (twice)

$\therefore y_c = (C_1 + C_2 x + C_3 x^2)e^{2x} + (C_4 + C_5 x)e^{2x}\cos x + (C_6 + C_7 x)e^{2x}\sin x.$

Minimal form of $y_p = (a_0 + a_1 x + a_2 x^2) \cdot x^2 \cdot e^{2x} \cdot \cos(x)$

$+ (b_0 + b_1 x + b_2 x^2) \cdot x^2 \cdot e^{2x} \cdot \sin(x)$

(b) $(D-0)^3(D+2)(D^2+4)^2 y = 0$. So $D = -2$, or $D = 0$ (thrice)

or $D = \pm 2i$ (twice each). Hence

$y_c = C_1 e^{-2x} + (C_2 + C_3 x + C_4 x^2)e^{0x} + (C_5 + C_6 x)\cos(2x) + (C_7 + C_8 x)\sin(2x)$

Now $(D-0)^3(D+2)(D^2+4)^2 y = 4x\cos^2(x) = 4x \cdot (1 + \cos(2x))/2$

$= 2x + 2x\cos(2x)$

Minimal form of $y_p = (C_0 + C_1 x) \cdot x^3 \cdot e^{0x} + (a_0 + a_1 x) \cdot x^2 \cdot \cos(2x)$

$+ (b_0 + b_1 x) \cdot x^2 \cdot \sin(2x).$

5. $y'' + y = \cot(x)$. Homog. ODE is $y'' + y = 0$. So

$(D^2+1)y = 0$. $\therefore D = \pm i$. So $y_c = A\cos x + B\sin x$.

Let $y_1 = \cos x$ and $y_2 = \sin x$. Then y_1 & y_2 are linearly indep.

and if we put $y_p = v_1 \cdot y_1 + v_2 \cdot y_2$ where

$$v_1' = \begin{vmatrix} 0 & \sin x \\ \frac{\cos x}{\sin x} & \cos x \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{0 - \cos x}{\cos^2 x + \sin^2 x} = \frac{-\cos x}{1} = -\cos(x)$$

$$\& \; v_2' = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{\cos x}{\sin x} \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{\frac{\cos^2 x}{\sin x} - 0}{1} = \frac{1 - \sin^2 x}{\sin x} = \csc x - \sin x,$$

then y_p will be a particular solution of $y'' + y = \cot x = \frac{\cos x}{\sin x}$.

$\therefore v_1' = -\sin x \Rightarrow v_1 = \int -\cos x\,dx = -\sin x$, and

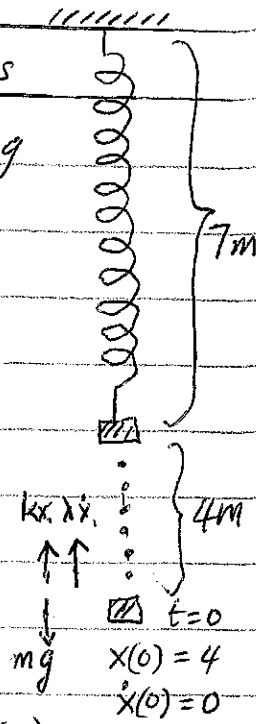
$v_2' = \int(\csc x - \sin x)\,dx = -\ln(\csc x + \cot x) + \cos x$

$\therefore y_p = \cos x(-\sin x) + \sin x[-\ln(\csc x + \cot x) + \cos x]$

$= -\sin x \cdot \ln(\csc x + \cot x)$ is a particular solution.

6. Let $x(t)$ = amount spring is extended at time t. Then

$m\ddot{x} = -kx - \lambda\dot{x} + mg$ because kx & $\lambda\dot{x}$ are forces working in the direction opposite to that of x while mg is working in the direction of increasing x.



$\therefore\quad 2\ddot{x} + 4\dot{x} + 20x = 2(10).$

$\therefore\quad \ddot{x} + 2\dot{x} + 10x = 10.$ The homog. ODE is

$\ddot{x} + 2\dot{x} + 10x = 0. \quad \therefore (D^2 + 2D + 10)x = 0$

$\therefore\quad D = (-2 \pm \sqrt{4-40})/2 = (-2 \pm 6i)/2 = -1 \pm 3i$

$\therefore\quad x_c = Ae^{-t}\cos(3t) + Be^{-t}\sin(3t).$ Also try

$x_p = a$. Then $\dot{x}_p = 0$ & $\ddot{x}_p = 0$. So $10a = 10$.

$\therefore\quad x_p = 1.$ So $x(t) = x_p(t) + x_c(t)$

$$= 1 + Ae^{-t}\cos(3t) + Be^{-t}\sin(3t)$$

$\therefore\quad \dot{x}(t) = 0 + A.(-1).e^{-t}\cos(3t) + A.(-3).e^{-t}\sin(3t)$

$$+ B.(3).e^{-t}\cos(3t) + B.(-1).e^{-t}\sin(3t)$$

$$= (3B - A)e^{-t}\cos(3t) - (3A + B).e^{-t}\sin(3t)$$

But $x(0) = 4$, so $4 = 1 + A.e^0.1 - B.0 \Rightarrow A = 3$

And $\dot{x}(0) = 0$, so $0 = (3B - A).e^0.1 - (3A+B).0 \Rightarrow B = 1$

$\therefore\quad x(t) = 1 + 3e^{-t}\cos(3t) + e^{-t}\sin(3t).$

END