MAP 2302 - Differential Equations — *Florida International University*

TEST 3 - Spring 2018 — *TIME: 75 min.*

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(15) 1. Find the general solution of the linear ODE $x^2.y'' + 2x.y' - 2.y = 10.x^2$ by first transforming it into a linear constant coefficient ODE in y and t.

(15) 2. Starting with $\mathcal{L}\{e^{at}\}(s) = 1/(s-a)$, use the properties of the Laplace transform to find (a) $\mathcal{L}\{sin(t)-1\}(s)$ and (b) $\mathcal{L}\{t.[sin(t) - 1]\}(s)$.

(15) 3. For each of the following ODEs, find the indicial equation and the form of two linearly independent Frobenius-series solution about $x_0 = 0$.

(a) $x^2.y'' + 2x.y' + (x^2 - 15/4).y = 0$.

(b) $x^2.y'' + 3x.y' + (7x + 2).y = 0$.

(20) 4. Solve each of the following *IVPs*, by using the Laplace transform.

(a) $y'(t) + 2.y(t) = -6.e^t$ with $y(0) = 1$.

(b) $y''(t) + 4.y'(t) + 5.y(t) = 0$ with $y(0) = 2$ & $y'(0) = -1$.

(15) 5. Solve the following system of linear ODEs, by using the Laplace transform.

$x'(t) - y(t) = 6,$ with $x(0) = 2$ and $y(0) = 0;$

$y'(t) - x(t) = 0.$

(20) 6. (a) Find the first 5 non-zero terms of the power series solution of the ODE $2.y'' - x.y' - y = 0$ with $y(0) = 4$ *and* $y'(0) = -6$.

(b) Define what it means for 0 to be a *singular point* and what it means for 0 to be a *regular singular point* of the ODE $y'' + P(x).y' + Q(x).y = 0$.

3(b) We have $x^2y'' + 3xy' + (7x+2).y = 0$. So the associated Cauchy-Euler ODE is $x^2y'' + 3xy' + 2y = 0$. Hence the aux. eq. is $\Delta(\Delta-1) + 3\Delta + 2 = 0$, $\therefore$ the indicial equation will be $r(r-1) + 3r + 2 = 0 \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = [-2 \pm \sqrt{4-4(2)}]/2 = -1 \pm i$

$\therefore$ two lin. indep. solutions will be $y_1(x) = x^{-1}.\cos(\ln x).\sum_{n=0}^{\infty} a_n.x^n$ and $y_2(x) = x^{-1}.\sin(\ln x).\sum_{n=0}^{\infty} b_n.x^n$ where $a_0 = 1$ & $b_0 = 1$.

4(a) $y'(t) + 2y(t) = -6.e^t$ and $y(0) = 1$.

$\therefore \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = -6.\mathcal{L}\{e^t\}$. So

$[s.\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\} = \frac{-6}{s-1} \Rightarrow (s+2)\mathcal{L}\{y\} = \frac{-6}{s-1} + y(0) = \frac{s-7}{s-1}$

$\therefore \mathcal{L}\{y\} = \frac{s-7}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} \Rightarrow s-7 = A(s+2) + B(s-1)$.

Putting $s=1$ gives us $1-7 = A(1+2)+0 \Rightarrow A = -2$

Putting $s=-2$ gives us $-2-7 = 0 + B(-2-1) \Rightarrow B = 3$.

$\therefore \mathcal{L}\{y\} = \frac{-2}{s-1} + \frac{3}{s+2} \Rightarrow y = -2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = -2e^t + 3e^{-2t}$.

(b) $y''(t) + 4.y'(t) + 5y(t) = 0$ and $y(0) = 2$ & $y'(0) = -1$.

So $\mathcal{L}\{y''\} + 4[\mathcal{L}\{y'\}] + 5.\mathcal{L}\{y\} = 0$.

$\therefore [s^2\mathcal{L}\{y\} - s.y(0) - y'(0)] + 4[\mathcal{L}\{y\} - y(0)] + 5.[\mathcal{L}\{y\}] = 0$

$\therefore (s^2 + 4s + 5).\mathcal{L}\{y\} = s.y(0) + y'(0) + 4y(0) = 2s - 1 + 8 = 2s + 7$

$\therefore \mathcal{L}\{y\} = \frac{2s+7}{(s+2)^2+1} = \frac{2(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}$. Hence

$y(t) = 2e^{-2t}.\cos(t) + 3e^{-2t}.\sin(t)$.

5. We have $\begin{cases} x'(t) - y(t) = 6 & (1) \\ y'(t) - x(t) = 0 & (2) \end{cases}$, with $x(0) = 2$ & $y(0) = 0$

From (1) $\mathcal{L}\{x'\} - \mathcal{L}\{y\} = \mathcal{L}\{6\}$. $\therefore [s.\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = 6/s$

$\therefore \mathcal{L}\{y\} = s.\mathcal{L}\{x\} - 2 - (6/s)$.

Now from (2) $\mathcal{L}\{y'\} - \mathcal{L}\{x\} = 0$. So $[s.\mathcal{L}\{y\} - y(0)] - \mathcal{L}\{x\} = 0$

$\therefore s.[s.\mathcal{L}\{x\} - 2 - (6/s)] - 0 - \mathcal{L}\{x\} = 0$

$\therefore (s^2 - 1).\mathcal{L}\{x\} = 2s + 6 \Rightarrow \mathcal{L}\{x\} = \frac{2s+6}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$

$\therefore 2s + 6 = A(s+1) + B(s-1)$.

MAD 2302 – Differential Equations   Florida Int'l Univ.

Solutions to Test #3   Spring 2018.

1. Let $x = e^t$. If we put $D = \frac{d}{dx}$ & $\Delta = \frac{d}{dt}$, then $xD = \Delta$ and $x^2D^2 = \Delta(\Delta - 1)$. So $x^2y'' + 2xy' - 2y = 10x^2$ becomes $(x^2D^2 + 2xD - 2)y = 10x^2$. $\therefore$ $[\Delta(\Delta-1) + 2\Delta - 2]\,y = 10.(e^t)^2$

$\therefore$ $(\Delta^2 + \Delta - 2)\,y = 10e^{2t}$. Homog. eq is $(\Delta^2 + \Delta - 2)\,y = 0$

$\therefore$ $(\Delta - 1)(\Delta + 2) = 0 \Rightarrow y_c = C_1 e^t + C_2 e^{-2t} = C_1 x^1 + C_2 x^{-2}$.

Try $y_p = A.e^{2t}$ (because RHS was $10.e^{2t}$ & 2 is not a root of the aux. eq.). Then $\dot{y}_p = 2Ae^{2t}$ & $\ddot{y}_p = 4Ae^{2t}$. So $(\Delta^2 + \Delta - 2)\,y = 10e^{2t}$ becomes $\ddot{y}_p + \dot{y}_p - 2y = 10e^{2t}$

$\therefore$ $4A.e^{2t} + 2Ae^{2t} - 2.Ae^{2t} = 10e^{2t} \Rightarrow 4A = 10 \Rightarrow A = 5/2$

$\therefore$ $y_p = (5/2)e^{2t}$. $\therefore$ $y = y_c + y_p = C_1 . x + C_2x^{-2} + (5/2).x^2$.

2. (a) We are given that $\mathcal{L}\{e^{at}\}(s) = 1/(s-a)$. So

$\mathcal{L}\{\sin(t) - 1\}(s) = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{1\} = \mathcal{L}\left\{\frac{e^{it} - e^{-it}}{2i}\right\} - \mathcal{L}\{e^{0t}\}$

$= \frac{1}{2i}\left[\mathcal{L}\{e^{it}\} - \mathcal{L}\{e^{-it}\}\right] - \frac{1}{s-0} = \frac{1}{2i}\left[\frac{1}{s-i} - \frac{1}{s+i}\right] - \frac{1}{s}$

$= \frac{1}{2i} \cdot \frac{2i}{s^2 - i^2} - \frac{1}{s} = \frac{1}{s^2+1} - \frac{1}{s}$.

(b) $\mathcal{L}\{t.[\sin(t) - 1]\}(s) = -\frac{d}{ds}\left[\mathcal{L}\{\sin(t) - 1\}\right] = -\frac{d}{ds}\left[(s^2+1)^{-1} - s^{-1}\right]$

$= -\left[(-1).(s^2+1)^{-2}.(2s) - (-1).s^{-2}\right] = \frac{2s}{s^2+1} - \frac{1}{s^2}$.

3 (a) We have $x^2y'' + 2x.y' + (x^2 - 15/4)y = 0$. So the associated Cauchy-Euler ODE is $x^2.y'' + 2x.y' - (15/4)y = 0$. So the aux. eq is $\Delta(\Delta-1) + 2\Delta - 15/4$. $\therefore$ indicial equation is

$r(r-1) + 2r - 15/4$. $\therefore$ $r^2 + r - 15/4 = 0$

$\therefore$ $r = \left[-1 \pm \sqrt{1 - 4(1)(-15/4)}\right]/2 = (-1 \pm \sqrt{16})/2 = (-1 \pm 4)/2$

$\therefore$ $r_1 = (-1+4)/2 = 3/2$ & $r_2 = (-1-4)/2 = -5/2$. So 2 lin. indep. sol are $y_1(x) = x^{3/2} \sum_{n=0}^{\infty} a_n . x^n$ & $y_2(x) = A.\,y_1(x).\ln x + x^{-5/2}\sum_{n=0}^{\infty} b_n . x^n$ where $a_0 = b_0 = 1$ & $A \in \mathbb{R}$ because $r_1 - r_2 = (3/2) - (-5/2) = 4 \in \mathbb{N}^+$.

Putting $s=1$ gives us $2(1)+6 = A(1+1) \Rightarrow A=4$. Putting $s=-1$ gives us $2(-1)+6 = B(-1-1) \Rightarrow B=-2$ $\therefore \mathcal{L}\{x\} = \frac{4}{s-1} - \frac{2}{s+1}$.

$\therefore x(t) = 4 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 4e^{t} - 2e^{-t}$. Also from (1)

$y(t) = -6 + x'(t) = -6 + 4e^{t} + 2e^{-t}$. <u>Check</u>: $x(0) = 4-2 = 2$ ✓
$y(0) = -6+4+2 = 0$ ✓

6 (a) Let $y = \sum_{n=0}^{\infty} a_n \cdot x^n$. Then $y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$ and

$y'' = \sum_{n=2}^{\infty} n(n-1) \cdot a_n \cdot x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n$. So $2y'' - xy' - y = 0$

becomes $2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \cdot \sum_{n=1}^{\infty} n a_n \cdot x^{n-1} - \sum_{n=0}^{\infty} a_n \cdot x^n \equiv 0$

$\therefore [2(0+2)(0+1) a_{0+2} - a_0] + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} - n \cdot a_n - a_n] x^n \equiv 0$

$\therefore 4a_2 = a_0$ and $2(n+2)(n+1) a_{n+2} = (n+1) a_n \Rightarrow a_{n+2} = \frac{a_n}{2(n+2)}$.

Now $y(0) = \sum_{n=0}^{\infty} a_0 \cdot x^0 = a_0 + 0 + 0 \cdots = a_0 \Rightarrow a_0 = 4$

And $y'(0) = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} = 1 \cdot a_1 + 0 + 0 + \cdots = a_1 \Rightarrow a_1 = -6$

Also $a_2 = a_0/4 = 4/4 = 1$, and putting

$n=1$ gives $a_3 = a_{1+2} = \frac{a_1}{2(1+2)} = \frac{-6}{6} = -1$

$n=2$ gives $a_4 = a_{2+2} = \frac{a_2}{2(2+2)} = \frac{1}{8}$. Thus first 5 non-zero terms are a follows.

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$
$= 4 - 6x + x^2 - x^3 + \frac{1}{8} x^4 + \ldots$

(b) $x_0 = 0$ is a <u>singular point</u> of $y'' + P(x) \cdot y' + Q(x) \cdot y = 0$ if at least one of the two functions $P_1(x)$ & $P_2(x)$ is not analytic at $x_0 = 0$. $x_0 = 0$ is a <u>regular</u> singular point of this ODE if $x_0 = 0$ is a singular point of it and both $(x-0) P(x)$ & $(x-0)^2 \cdot Q(x)$ are analytic at $x_0 = 0$. END.