*MAP 2302 - Differential Equations* | *Florida International University*
*TEST #1 - Spring 2022* | *TIME: 75 min.*

*Answer all 6 questions.* **No calculators, formula sheets, or cell phones are allowed.** *An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible.* **Begin each of the 6 questions on 6 separate pages.**

(15) 1 (a) Define what it means for the *ODE* $dy/dx = f(x,y)$ to be *homogeneous*.
(b) Find the *general solution* of the *ODE* $(x^2y + 2y^3).dx - xy^2.dy = 0$.

(15) 2. Check that $\{2xy.\cos(x^2y) - 1/x^2\}.dx + \{x^2.\cos(x^2y) + e^{y/2}\}.dy = 0$
(a) is an *exact ODE,* and (b) find its *general solution.*

(15) 3 (a) Define what is *an integrating factor of the non-exact ODE* $M.dx + N.dy = 0$.
(b) Find *the* solution of the ODE $dy/dx - y = 6x.e^x$ with $y(1) = 0$.

(15) 4. Find the *general solution* to the differential equation $dy/dx - y/x = x.y^{-2}$.

(20) 5. The population of a colony of micro-organisms satisfy the logistic equation $dP/dt = -P.[(P/4) - 1]$, where $t$ is measured in *hours* and $P$ in *millions*.
(a) If $P(0) = 6$ *million,* find the *population* after $t$ hours has elapsed.
(b) How *many hours* will it take for the *population* to reach *5 million.*

(20) 6. A ball of mass *2 kg* thrown vertically upwards from sea level with velocity *15* $ms^{-1}$. If $g = 10\ ms^{-2}$ and the *air resistance* is $\lambda v$, where $\lambda = 4\ kgs^{-1}$, find
(a) the time it takes for the ball to reach its greatest height, and
(b) the greatest height above sea-level that the ball reaches.

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Solutions to Test #1 Spring 2022

1 (a) $\frac{dy}{dx} = f(x,y)$ is a homogeneous ODE if we can find a function $g$ such that $f(x,y) = g(y/x)$.

(b) $(x^2y + 2y^3)dx - (xy^2)dy = 0$ So $(xy^2)dy = (x^2y + 2y^3)dx$

$\therefore \frac{dy}{dx} = \frac{x^2y+2y^3}{xy^2} = \frac{x}{y} + 2\left(\frac{y}{x}\right) = \frac{1}{v} + 2v$ if $v = \frac{y}{x}$

Now $y = xv$, so $\frac{dy}{dx} = 1.v + x.\frac{dv}{dx}$. Thus

$v + x\frac{dv}{dx} = \frac{1}{v} + 2v \Rightarrow x\,dv = \left(\frac{1}{v} + v\right)dx = \frac{v\,dx}{v^2+1}$

$\therefore \frac{2v}{v^2+1}dv = \frac{2}{x}dx$. So $\int \frac{2v\,dv}{v^2+1} = \int \frac{2}{x}dx$.

$\therefore \ln(v^2+1) = 2\ln(x) + C \quad \therefore \ln(v^2+1) - \ln(x^2) = C$

and so $\ln[(v^2+1)/x^2] = C \Rightarrow \frac{v^2+1}{x^2} = e^C = A$ (say)

$\therefore v^2 + 1 = Ax^2$ and so $(y/x)^2 + 1 = Ax^2 \Rightarrow y^2 = (Ax^2-1)x^2$.

2 (a) Let $M = 2xy.\cos(x^2y) - x^{-2}$ & $N = x^2.\cos(x^2y) + e^{y/2}$. Then

$\frac{\partial M}{\partial y} = 2x.1.\cos(x^2y) + 2xy.x^2.[-\sin(x^2y)] - 0$ and

$\frac{\partial N}{\partial x} = 2x.\cos(x^2y) + x^2.(2xy).[-\sin(x^2y)] + 0$. So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore M\,dx + N\,dy = 0$ is an exact ODE.

(b) $\frac{\partial F}{\partial x} = M$, so $F(x,y) = \int M(x,y).\partial x = \int \cos(x^2y).(2xy.\partial x) - x^{-2}\partial x$

Put $u = x^2y$. Then $\partial u = 2xy\,\partial x$ and $\cos(x^2y) = \cos(u)$

$= \int \cos(u).\partial u - \int x^{-2}\partial x$

$= \sin(u) - x^{-2+1}/(-2+1) + \varphi(y)$

$= \sin(x^2y) + 1/x + \varphi(y)$

$\therefore \frac{\partial F}{\partial y} = x^2\cos(x^2y) + 0 + \varphi'(y)$

But $\frac{\partial F}{\partial y} = N = x^2\cos(x^2y) + e^{y/2}$. So $\varphi'(y) = e^{y/2} \Rightarrow \varphi(y) = 2e^{y/2} + C_1$

Hence $F(x,y) = \sin(x^2y) + (1/x) + 2e^{y/2} + C_1$. But $dF = M\,dx + N\,dy = 0$

So $F(x,y) = C_2$. $\therefore \sin(x^2y) + (1/x) + 2e^{y/2} = C_1 - C_2 = C$ is gen. sol.

3 (a) An integrating factor of the non-exact ODE $M\,dx + N\,dy = 0$ is any function $\mu = \mu(x,y)$ such that $\mu.M\,dx + \mu.N\,dy = 0$ is an exact ODE. This means $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$.

(b) $\frac{dy}{dx} - y = 6x.e^x$. This a linear ODE, so I.F. $= e^{\int(-1)dx} = e^{-x}$.

$\therefore e^{-x}.\frac{dy}{dx} - e^{-x}.y = 6x.e^x.e^{-x} \Rightarrow \frac{d}{dx}(e^{-x}.y) = 6x$. So

3(b) $e^{-x}.y = \int 6x.dx = 3x^2+C$ $\therefore$ $y = (3x^2+C).e^{x}$.

But $y(1)=0$. So $0 = [3(1)^2+C].e^{1} \Rightarrow C=-3$. $\therefore$ $y = 3(x^2-1).e^{x}$.

4(a) $\frac{dy}{dx} - \frac{y}{x} = x.y^{-2}$. This is a Bernoulli ODE with $\alpha=-2$.

So multiply both sides by $(1-\alpha)y^{-\alpha} = [1-(-2)].y^{-(-2)} = 3y^2$ and put $v = y^{1-\alpha} = y^{1-(-2)} = y^3$. Then we will get $\frac{dv}{dy} = 3y^2$

$3y^2\frac{dy}{dx} - 3y^2.\frac{y}{x} = x.3y^2.y^{-2} \Rightarrow 3y^2\frac{dy}{dx} - \frac{3y^3}{x} = 3x$.

So $\frac{dv}{dy}.\frac{dy}{dx} - 3\frac{v}{x} = 3x \Rightarrow \frac{dv}{dx} - \frac{3}{x}.v = 3x$.

(b) If we did this correctly we should have gotten a linear ODE — and we did. So $I.F. = e^{\int(-3/x)dx} = e^{-3\ln x} = e^{\ln(x^{-3})} = \frac{1}{x^3}$.

$\therefore$ $\frac{1}{x^3}.\frac{dv}{dx} - \frac{1}{x^3}.\frac{3}{x}.v = \frac{1}{x^3}.3x$. $\therefore \int\frac{d}{dx}\left(\frac{1}{x^3}.v\right) = \int\frac{3}{x^2} = \frac{-3}{x}+C$

$\therefore$ $v/x^3 = (-3/x)+C \Rightarrow y^3 = v = x^3(C-3/x) = Cx^3-3x^2$.

5(a) $dP/dt = -P[(P/4)-1] = -P(P-4)/4$.

$\therefore$ $\frac{-4\,dP}{P(P-4)} = dt$

Put $\frac{-4}{P(P-4)} = \frac{A}{P} + \frac{B}{P-4}$
Then $-4 = A(P-4)+BP$
Putting $P=0$ gives $-4=-4A$, so $A=1$
putting $P=4$ gives $-4=4B \Rightarrow B=-1$

$\therefore$ $\frac{1}{P} - \frac{1}{P-4} = dt$

$\therefore$ $\int\frac{1}{P} - \frac{1}{P-4} = dt$

So $\ln(P) - \ln(P-4) = t+C$ $\therefore$ $\ln\left(\frac{P}{P-4}\right) = t+C$.

But $P(0)=6$. So $\ln\left(\frac{6}{6-4}\right) = 0+C \Rightarrow C = \ln(3)$.

$\therefore$ $\ln\left(\frac{P}{P-4}\right) = t+\ln 3 \Rightarrow \ln\left[\frac{P}{3(P-4)}\right] = t \Rightarrow \frac{P}{3(P-4)} = e^{t}$

$\therefore$ $\frac{3(P-4)}{P} = e^{-t} \Rightarrow 3P-12 = Pe^{-t} \Rightarrow P(3-e^{-t}) = 12$

$\therefore$ $P(t) = 12/(3-e^{-t})$ million. [Check: $P(0) = 12/(3-e^{0}) = 6$ ✓]

(b) From part (a), $t = \ln\left[\frac{P}{3(P-4)}\right]$. So when $P=5$, then $t = \ln\left[\frac{5}{3(5-4)}\right] = \ln(5/3)$ hours.

6(a) From Newton's 2nd law we know $m\frac{dv}{dt} = -mg-\lambda v$.

So $2\frac{dv}{dt} = -2(10)-4v \Rightarrow \frac{dv}{dt} = -2(5+v)$, so $\frac{dv}{v+5} = -2dt$.

$\therefore \int dv/(v+5) = -\int 2dt \Rightarrow \ln(v+5) = -2t+C_1$. B $v(0)=15$, so

6(a) So $\ln(15+5) = z(0) + C_1 \Rightarrow C_1 = \ln(20)$.

$\therefore \ln(v+5) = -2t + \ln(20)$. So $2t = \ln(20) - \ln(v+5)$

$\therefore t = \frac{1}{2}\ln\left(\frac{20}{v+5}\right)$. The ball will reach its greatest height when $v(t)=0$. This will happen when

$$t = \frac{1}{2}\ln\left(\frac{20}{0+5}\right) = \frac{1}{2}\ln(4) = \frac{1}{2}\ln(2^2) = \frac{1}{2}\cdot 2 \cdot \ln(2) = \ln(2) \text{ seconds}$$

(b) We know that $\ln(v+5) = -2t + \ln(20)$.

$\therefore \ln\left(\frac{v+5}{20}\right) = -2t \Rightarrow \frac{v+5}{20} = e^{-2t}$

$\therefore v = 20e^{-2t} - 5$. $\therefore \frac{dz}{dt} = 20e^{-2t} - 5$

$\therefore z(t) = \int (20e^{-2t} - 5)\,dt = -10e^{-2t} - 5t + C_2$

But $z(0) = 0$. So $0 = -10.e^0 - 5(0) + C_2 \Rightarrow C_2 = 10$

$\therefore z(t) = 10 - 10e^{-2t} - 5t$.

The ball reaches its greatest height when $t = \ln(2)$.
So this height will be

$$\begin{aligned} z(\ln 2) &= 10 - 10.e^{-2\ln(2)} - 5\ln(2) \\ &= 10 - 10.e^{\ln(2^{-2})} - 5\ln(2) \\ &= 10 - \frac{10}{4} - 5\ln 2 = 5\left[\frac{3}{2} - \ln(2)\right] \text{ metres.} \end{aligned}$$

END.

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