MAP 2302 - Differential Equations | *Florida International University*
TEST #2 - Spring 2022 | *TIME: 75 min.*

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(20) 1. Find the solution of each of the following homogeneous differential equations.
(a) $y'' - 2y' - 3y = 0$ with $y(0) = 3$ and $y'(0) = 5$.
(b) $y'' + 9y = 0$ with $y(0) = 2$ and $y'(0) = 3$.

(15) 2. (a) Define what it means for the functions $f_1(x), f_2(x), f_3(x)$ to be ***linearly dependent.***
(b) Find the *general solution* of the differential equation $y'' + 2y' - 3y = 6.e^x$.

(15) 3. Find the *general solution* of the ODE $y'' + 2y' + 2y = 5.\cos(x)$.

(20) 4. Find y_c and give *the minimal form* of the y_p that one should try, for each of the following non-homogeneous linear differential equations.
(a) $\{(D-1)^3 . (D^2 + 4D + 5)^2\} y = x.e^x + e^{-2x}.\cos(x)$.
(b) $\{D . (D+2)^2 . (D^2 + 4)^2\} y = 6 . x . \sin^2(x)$.

(15) 5. Find *a particular solution*, y_p, of the *ODE* $y'' + 2y' + y = x^{-2}. e^{-x}$ by using the method of *variation of parameters*.

(15) 6. A body of mass *3kg* is attached to a *Hooke-type* spring and suspended from a very high ceiling. The natural length of the spring is *5m*, the spring constant *k* is $6\ Nm^{-1}$ and the air resistance is λv where $\lambda = 6\ Nsm^{-1}$. If the spring is stretched by an amount of *2m,* and then set loose from rest at time t = 0, find the *amount, x(t),* it will be extended at all subsequent times. [*Use acceleration due to gravity,* $g = 10\ ms^{-2}$]

MAP 2302 - Differential Equations Florida Int'l Univ.
Solutions to Test #2 Spring 2022.

1(a) $y'' - 2y' - 3y = 0 \Rightarrow (D^2 - 2D - 3)y = 0$. $\therefore (D+1)(D-3) = 0$.
$\therefore y = Ae^{-x} + Be^{3x}$. $y(0) = 5 \Rightarrow A + B = 3 \Rightarrow A = 3 - B$.
& $y' = -Ae^{-x} + 3Be^{3x}$. $y'(0) = 3 \Rightarrow -A + 3B = 5 \Rightarrow -(3-B) + 3B = 5$
$\therefore 4B = 8 \Rightarrow B = 2$. $\therefore A = 3 - 2 = 1$. $\therefore y(x) = e^{-x} + 2e^{3x}$

(b) $y'' + 9y = 0 \Rightarrow (D^2 + 9)y = 0$ $\therefore (D - 3i)(D + 3i) = 0$
$\therefore y = A\cos(3x) + B\sin(3x)$ $y(0) = 2 \Rightarrow A.1 + B.0 = 2 \Rightarrow A = 2$
So $y' = -3A\sin(3x) + 3B\cos(3x)$ $y'(0) = 3 \Rightarrow -A.0 + 3B.1 = 3 \Rightarrow B = 1$
$\therefore y(x) = 2\cos(3x) + \sin(3x)$. Check $y(0) = 2$, $y'(0) = 3$ ✓

2(a) The functions $f_1(x)$, $f_2(x)$, & $f_3(x)$ are linearly dependent if we can find constants $\alpha_1, \alpha_2, \alpha_3$ with at least one of them being non-zero such that $\alpha_1 . f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) \equiv 0$.

(b) $y_c'' + 2y_c' - 3y_c = 0 \Rightarrow (D^2 + 2D - 3)y_c$. $\therefore (D-1)(D+3) = 0$
$\therefore y_c = C_1 e^{x} + C_2 e^{-3x}$. To find a particular solution of $y'' + 2y' - 3y = 6e^{x}$ we must try $y_p = b.x^1.e^{x}$ because $D = 1$ is a root of the auxiliary eq. of multiplicity 1. So

$y_p = bxe^{x}$, $y_p' = b.1.e^{x} + bx.e^{x} = (b + bx)e^{x}$, and
$y_p'' = (0 + b).e^{x} + (b + bx).e^{x} = (2b + bx)e^{x}$.

So the non-homogeneous equation $y_p'' + 2y_p' - 3y_p = 6e^{x}$ becomes $(2b + bx)e^{x} + 2(b + bx)e^{x} - 2bx.e^{x} = 6e^{x} \Rightarrow 4be^{x} = 6e^{x}$.
$\therefore b = 3/2$. $\therefore y = y_c + y_p = C_1 e^{x} + C_2 e^{-3x} + (3/2).x.e^{x}$.

3. Homog. Eq. is $y_c'' + 2y_c' + 2y_c = 0$. $\therefore D^2 + 2D + 2 = 0$. So
$D = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$. $\therefore y_c = e^{-x}(C_1\cos x + C_2 \sin x)$
Since the RHS of the ODE is $5\cos x$, try $y_p = A\cos x + B\sin x$
Then $y_p' = -A\sin x + B\cos x$ & $y_p'' = -A\cos x - B\sin x$. Thus
$y_p'' + 2y_p' + 2y_p = 5\cos x \Rightarrow [-A + 2(B) + 2A]\cos x + [-B - 2A + 2B]\sin x = 5\cos x$.

$\therefore\ A+2B=5\ \&\ B-2A=0.\ \therefore\ B=2A\ \&\ \text{so}\ A+2(2A)=5 \Rightarrow A=1$

$\therefore\ B=2.$ So $y = y_c + y_p = (C_1\cos x + C_2 \sin x)e^{-x} + \cos x + 2\sin x.$

4(a) $\{(D-1)^3(D^2+4D+5)^2\}=0 \Rightarrow D=1$ (3 times) or $D=-2\pm i$ (twice)

$\therefore\ y_c = (C_1+C_2x+C_3x^2)e^x + (C_4+C_5x)e^{-2x}\cos x + (C_6+C_7x)e^{-2x}\sin x.$

Minimal form of y_p that we should try should be

$$y_p = (a_0+a_1x).x^3.e^x + (b_0).x^2.e^{-2x}\cos x + (c_0).x^2.e^{-2x}\sin x$$

because $x.e^x$ = (a polynomial of degree 1) times e^x & mult(1)=3

and $1.e^{-2x}\cos x$ = (a polynomial of deg. 1) . $e^{-2x}.\cos x$ & mult$(-2\pm i)=2$.

(b) $\{(D-0)(D+2)^2(D^2+4)^2\}=0 \Rightarrow D=0,\ D=-2$ (twice) & $D=\pm 2i$ (twice)

$\therefore\ y_c = C_1.e^{0x} + (C_2+C_3x).e^{-2x} + (C_4+C_5x)\cos(2x) + (C_6+C_7x)\sin(2x).$

Minimal form of y_p that we should try should be

$$y_p = (a_0+a_1x).x^1.e^{0x} + (b_0+b_1x).x^2.\cos(2x) + (c_0+c_1x).x^2.\sin(2x)$$

because $RHS = 6x.\sin^2 x = 6x.\frac{1}{2}(1-\cos 2x) = 3x - 3x.\cos(2x)$

and $3x = 3x.e^{0x}$ = (polynomial of deg. 1). e^{0x} & multiplicity(0)=1

and $3x\cos(2x)$ = (polynomial of deg. 1). $\cos(2x)$ & mult$(\pm 2i)=2$.

5. Homog. Eq. is $y_c'' + 2y_c' + y_c = 0.\ \therefore\ D^2+2D+1=(D+1)^2=0$

$\therefore\ y_c = C_1e^{-x} + C_2.x.e^{-x}$, So take $y_1 = e^{-x}$ & $y_2 = xe^{-x}$.

Then $y_1' = -e^{-x}$ and $y_2' = 1.e^{-x} - x.e^{-x} = (1-x)e^{-x}$. So

we know there will be a particular solution $y_p = v_1y_1 + v_2y_2$

where $v_1' = \begin{vmatrix} 0 & y_2 \\ x^{-2}e^{-x} & y_2' \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{-x} \\ x^{-2}e^{-x} & (1-x)e^{-x} \end{vmatrix} \Big/ \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$

$= -x^{-1}e^{-2x}/[(1-x)e^{-2x} + xe^{-2x}] = -x^{-1} \Rightarrow v_1 = \int -x^{-1}dx = -\ln(x).$

& $v_2' = \begin{vmatrix} y_1 & 0 \\ y_1' & x^{-2}e^{-x} \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & x^{-2}e^{-x} \end{vmatrix} \Big/ (e^{-2x}) = \dfrac{x^{-2}e^{-2x}}{e^{-2x}} = x^{-2}$

$\therefore\ v_2 = \int x^{-2}dx = x^{-2+1}/(-2+1) = -(x^{-1}).$

$\therefore\ y_p = -\ln(x).e^{-x} - (x^{-1}).xe^{-x} = -(1+\ln x).e^{-x}.$

6. We have $m\ddot{x} = -k\dot{x} - \lambda x + mg$.

$\therefore m\ddot{x} + k\dot{x} + \lambda x = mg$

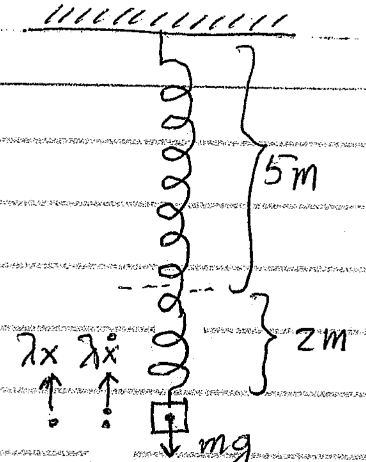
$\therefore 3\ddot{x} + 6\dot{x} + 6x = 3(10)$

$\therefore \ddot{x} + 2\dot{x} + 2x = 10 \quad (*)$

$(\Delta^2 + 2\Delta + 2)x = 0 \quad \left(\text{Here } \Delta = \frac{d}{dt}\right)$

$\Rightarrow \Delta = -1 \pm i$. So

$x_c(t) = e^{-t}[A\cos t + B\sin t]$.



$x(0) = 2, \ \dot{x}(0) = 0$

Try $x_p(t) = b$ because RHS $= 10$.

Then $\dot{x}_p = 0$ & $\ddot{x}_p = 0$. So the ODE $(*)$ becomes $\ddot{x}_p + 2\dot{x}_p + 2x_p = 10 \Rightarrow 0 + 0 + 2x_p = 10$.

$\therefore x_p = 5$. $\therefore x(t) = x_c(t) + x_p(t) = 5 + e^{-t}[A\cos t + B\sin t]$

So $\dot{x}(t) = e^{-t}[-A\sin t + B\cos t] - e^{-t}[A\cos t + B\sin t] + 0$.

$\therefore x(0) = 2 \Rightarrow 5 + e^{-0}[A.1 + B.0] = 2 \Rightarrow A = -3$.

and $\dot{x}(0) = 0 \Rightarrow e^{-0}[-A.0 + B.1] - e^{-0}[A.1 + B.0] = 0$

$\Rightarrow B - A = 0 \Rightarrow B = A = -3$.

$\therefore x(t) = 5 + e^{-t}[-3\cos t - 3\sin t] = 5 - 3e^{-t}[\cos t + \sin t]$

END

Check: Extras

$$\begin{cases} x(0) = 5 - 3e^{-0}[1+0] = 5 - 3 = 2 \checkmark \\ \dot{x}(t) = 0 + 3e^{-t}[\cos t + \sin t] - 3e^{-t}[-\sin t + \cos t] \\ \dot{x}(0) = 0 + 3.[1+0] - 3.[-0+1] = 3 - 3 = 0 \checkmark \end{cases}$$

By the way at time $t = 0$, $\dot{x} = 0$ because the weight is released from rest. Also $\ddot{x}(0) + 2\dot{x}(0) + 2x(0) = 10$, so $\ddot{x}(0) = 10 - 2\dot{x}(0) - 2x(0) = 10 - 0 - 2(2) = 6 \ ms^{-2}$.

This means the body is accelerating downwards the moment it is released (since x is measured in the downward direction).

Also $x(t) = 5 - 3.e^{-t}.\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right]$

$= 5 - (3\sqrt{2})e^{-t}\left[\cos t . \cos\frac{\pi}{4} + \sin t . \sin\frac{\pi}{4}\right]$

$= 5 - 3\sqrt{2}.e^{-t}\cos(t - \pi/4)$, in case you're asked for this.