MAP 2302 - Differential Equations | *Florida International University*
TEST 3 - Spring 2022 | *TIME: 75 min.*

Answer all 6 questions. No calculators, formula sheets, or cell phones are allowed. An unjustified answer will receive little or no credit. Show all working and simplify your answer as far as possible. ***Begin each of the 6 questions on 6 separate pages.***

(15) 1. Find the general solution of the linear ODE $x^2.y'' - x.y' - 3.y = 12x$ by first transforming it into a linear constant coefficient ODE in y and t.

(12) 2. Starting with $\mathscr{L}\{e^{at}\}(s) = 1/(s-a)$, use the properties of the Laplace transform to find (a) $\mathscr{L}\{\cos(3t)\}(s)$ and (b) $\mathscr{L}\{t.[\cos(3t)]\}(s)$.

(20) 3. Solve each of the following *IVPs*, by using the Laplace transform.

(a) $y'(t) + 2.y(t) = -6.e^t$ with $y(0) = 4$.

(b) $y''(t) - 2.y'(t) + 2.y(t) = 0$ with $y(0) = 1$ & $y'(0) = 3$.

(15) 4. Solve the following *system* of linear ODEs, by using the Laplace transform.

$x'(t) - y(t) = 5$ *and* $y'(t) - x(t) = 0;$ with $x(0) = 1$ and $y(0) = 0.$

(18) 5. For each of the following ODEs, find *the indicial equation* and the *form* of two linearly independent *Frobenius-series solution* about $x_0 = 0$.

(a) $x^2.y'' + 2x.y' + (2x^3 + 5/4).y = 0.$

(b) $x^2.y'' - x.y' + (5x^2 + 3/4).y = 0.$

(20) 6. (a) Find the first 5 non-zero terms of the power series solution of the ODE

$y'' - x.y' - 2.y = 0$ with $y(0) = 2$ *and* $y'(0) = -3.$

(b) Define what it means for *0* to be a *singular point* and what it means for *0* to be a *regular singular point* of the ODE $y'' + P(x).y' + Q(x).y = 0.$

MAP 2302 - Differential Equations Florida International Univ.

Solutions to Test # 3 Spring 2022

1. $x^2y'' - xy' - 3y = 12x$. Putting $x = e^t$, we get $xD = \Delta$ & $x^2D = \Delta(\Delta-1)$

So $[\Delta(\Delta-1) - \Delta - 3]y = 12e^t$. $\therefore (\Delta^2 - 2\Delta - 3)y = 0$ will be the homog. equation. $(\Delta+1)(\Delta-3)y = 0 \Rightarrow \Delta = -1$ or 3.

$\therefore y_c = Ae^{-t} + Be^{3t}$. Try $y_p = a.e^t$ (since 1 is not a root of the auxiliary equation). Then $\dot{y}_p = ae^t$ and $\ddot{y}_p = ae^t$. So $\ddot{y} - 2\dot{y} - 3y = 12e^t$ becomes

$ae^t - 2ae^t - 3ae^t = 12e^t$. $\therefore -4a = 12 \Rightarrow a = -3$.

$\therefore y_p = -3e^t$. So $y = Ae^{-t} + Be^{3t} - 3e^t = Ax^{-1} + Bx^3 - 3x$.

2(a) $\mathcal{L}\{\cos(3t)\} = \mathcal{L}\{\frac{1}{2}(e^{3it} + e^{-3it})\} = \frac{1}{2}\left[\mathcal{L}\{e^{3it}\} + \mathcal{L}\{e^{-3it}\}\right]$

$= \frac{1}{2}\left(\frac{1}{s-3i} + \frac{1}{s+3i}\right) = \frac{1}{2}\,\frac{(s+3i)+(s-3i)}{(s-3i)(s+3i)} = \frac{s}{s^2+9}$.

(b) $\mathcal{L}\{t.\cos(3t)\} = -\frac{d}{ds}\left[\mathcal{L}\{\cos(3t)\}\right] = -\frac{d}{ds}\left[s.(s^2+9)^{-1}\right]$

$= -\left[1.(s^2+9)^{-1} + s.(-1).(s^2+9)^{-2}.(2s)\right] = \frac{2s^2 - (s^2+9)}{(s^2+9)^2}$

$= (s^2-9)/(s^2+9)^2$.

3(a) $y'(t) + 2y(t) = -6e^t$. $\therefore \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = -6.\mathcal{L}\{e^t\}$

$\therefore [s.\mathcal{L}\{y\} - y(0)] + 2[\mathcal{L}\{y\}] = -6/(s-1)$.

$\therefore (s+2)\mathcal{L}\{y\} = y(0) - 6/(s-1) = 4 - 6/(s-1) = \frac{4s-10}{s-1}$

$\therefore \mathcal{L}\{y\} = \frac{4s-10}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} = \frac{A(s+2) + B(s-1)}{(s-1)(s+2)}$

$\therefore A(s+2) + B(s-1) = 4s - 10$. Putting $s=1$ gives $3A = -6 \Rightarrow A = -2$

Putting $s=-2$ gives $-3B = -18 \Rightarrow B = 6$

$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{-2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s+2}\right\} = 6e^{-2t} - 2e^t$.

(b) $y''(t) - 2y'(t) + 2y(t) = 0$. $\therefore \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0$

$\therefore [s^2.\mathcal{L}\{y\} - s.y(0) - y'(0)] - 2[s\mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = 0$

$\therefore [s^2 - 2s - 2].\mathcal{L}\{y\} = s.y(0) + y'(0) - 2.y(0) = s.1 + 3 - 2(1) = s+1$

$\therefore \mathcal{L}\{y\} = (s+1)/[(s-1)^2+1] = (s-1)/[(s-1)^2+1] + 2/[(s-1)^2+1]$

3(b) $\therefore\ y(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+1}\right\} = e^t\cos t + 2e^t\sin t.$

4. $x'(t) - y(t) = 5$ ① $\therefore\ \mathcal{L}\{x'\} - \mathcal{L}\{y\} = \mathcal{L}(5) = 5/s$

$y'(t) - x(t) = 0$ ② and $\mathcal{L}\{y'\} - \mathcal{L}\{x\} = 0$

$\therefore\ s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{x\} = 0 \Rightarrow \mathcal{L}\{x\} = s\,\mathcal{L}\{y\}$

and $s.\mathcal{L}\{x\} - x(0) - \mathcal{L}\{y\} = 5/s \Rightarrow s.\mathcal{L}\{x\} - \mathcal{L}\{y\} = \frac{5}{s}+1 = \frac{s+5}{s}$

Substituting $s.\mathcal{L}\{y\}$ for $\mathcal{L}\{x\}$, we get

$s.s\mathcal{L}\{y\} - \mathcal{L}\{y\} = (s+5)/s \quad \therefore\ (s^2-1)\,\mathcal{L}\{y\} = (s+5)/s$

$\therefore\ \mathcal{L}\{y\} = \frac{s+5}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} = \frac{A(s-1)(s+1) + B.s(s+1) + Cs(s-1)}{s(s-1)(s+1)}$

Putting $s=0$ gives $A(-1) = 5 \Rightarrow A = -5$

Putting $s=1$ gives $B(2) = 6 \Rightarrow B = 3$

Putting $s=-1$ gives $C(2) = 4 \Rightarrow C = 2$.

$\therefore\ \mathcal{L}\{y\} = \frac{-5}{s} + \frac{3}{s-1} + \frac{2}{s+1} \Rightarrow y(t) = -5 + 3e^t + 2e^{-t}$

From ② $x(t) = y'(t) = 3e^t - 2e^{-t}$. <u>Check</u>: $x(0)=1$, $y(0)=0$.

5(a) $x^2y'' + 2xy' + (2x^3 + 5/4)y = 0$. The corresp. Cauchy-Euler ODE is $x^2y'' + 2xy' + (5/4)y = 0$. $\therefore\ [\Delta(\Delta-1) + 2\Delta + 5/4]y = 0$ where $xD = \Delta$ & $x^2D = \Delta(\Delta-1)$. So indicial equation will be

$r(r-1) + 2r + (5/4) = 0 \quad \therefore\ r^2 + r + 5/4 = 0, \quad r = (-1 \pm \sqrt{1-5})/2 = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$

$\therefore\ r_1 = -\frac{1}{2} + i$ & $r_2 = -\frac{1}{2} - i$ (complex roots)

So two lin. independent series solutions will be

$y_1(x) = x^{-1/2}\cos(\ln x).\sum_{n=0}^{\infty} a_n x^n$ & $y_2(x) = x^{-1/2}.\sin(\ln x)\sum_{n=0}^{\infty} b_n x^n$ with $a_0 = b_0 = 1$.

(b) $x^2y'' - xy' + (5x^2 + 3/4)y = 0$. Cauchy-Euler: $x^2y'' - xy' + \frac{3}{4}y = 0$

$\therefore\ [\Delta(\Delta-1) - \Delta + (3/4)]y = 0 \quad \therefore\ r(r-1) - r + 3/4 = 0$ (indicial eq.)

$\therefore\ r^2 - 2r + 3/4 = 0$

$\therefore\ r = (2 \pm \sqrt{4-3})/2 = (2 \pm 1)/2 = 1 \pm 1/2 = 3/2$ or $1/2$. $r_1 = \frac{3}{2}$, $r_2 = \frac{1}{2}$

Since $r_1 - r_2 = 1$ is a positive integer, 2 lin. independent series solutions will be $y_1(x) = x^{3/2}\sum_{n=0}^{\infty} a_n.x^n$ with $a_0 = 1$, and

$y_2(x) = A.y_1(x).\ln(x) + x^{1/2}\sum_{n=0}^{\infty} b_n.x^n$ with $b_0 = 1$. (Here A may or may not be 0.)

6(a) Let $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$

Then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots + (n+1).a_{n+1} x^n + \dots$

So $y(0) = a_0 + 0 + 0 + \dots = a_0 \Rightarrow a_0 = 2$ bec. $y(0) = 2$

and $y'(0) = a_1 + 0 + 0 + \dots = a_1 \Rightarrow a_1 = -3$ bec. $y'(0) = -3$.

Now $y'' - xy' - 2y = 0$

$$\begin{aligned} \therefore 0 = &\ 2a_2 + 2(3)a_3 x + 4(3)a_4 x^2 + \dots + (n+2)(n+1)a_{n+2} x^n + \dots \\ & - a_1 x - 2 a_2 x^2 - \dots - n.a_n.x^n - \dots \\ & - 2a_0 - 2a_1 x - 2a_2 x^2 - \dots - 2.a_n.x^n - \dots \end{aligned}$$

$\therefore$ $2a_2 - 2a_0 = 0 \Rightarrow 2a_2 = 2a_0 \Rightarrow a_2 = a_0 = 2$

$6a_3 - a_1 - 2a_1 = 0 \Rightarrow 6a_3 = 3a_1 \Rightarrow a_3 = -9/6 = \frac{-3}{2}$

$12a_4 - 2a_2 - 2a_2 = 0 \Rightarrow 12a_4 = 4a_2 \Rightarrow a_4 = 4(2)/12 = 2/3$

$\therefore y(x) = 2 - 3x + 2x^2 - \frac{3}{2}x^3 + \frac{2}{3}x^4 + \dots$

[Extra: In general $(n+2)(n+1)a_{n+2} = (n+2)a_n$.

So $a_{n+2} = [2(n+1)/(n+2)(n+1)]a_n = [1/(n+1)]a_n$.

Putting $n=0$ gives $a_2 = [1/(0+1)]a_0 = a_0 = 2$

Putting $n=1$ gives $a_3 = [1/(1+1)]a_1 = \frac{1}{2}a_1 = -\frac{3}{2}$

Putting $n=2$ gives $a_4 = [1/(2+1)]a_2 = \frac{1}{3}a_2 = \frac{2}{3}$.]

(b) $x_0 = 0$ is a <u>singular point</u> of the ODE $y'' + P(x).y' + Q(x)y = 0$ if at least one of the two functions $P(x)$ & $Q(x)$ is not analytic at $x_0 = 0$. $x_0 = 0$ is a <u>regular</u> singular point of this ODE is if it is a singular point of it and if both $(x-0).P(x)$ & $(x-0)^2 Q(x)$ are analytic at $x_0 = 0$. END