MAP 2302 – Differential Equations — Florida Int'l Univ.

Test # 1 – Spring 2006 — TIME : 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. No calculators or formula sheets are allowed.

(15) 1. Find the unique solution of the ODE

$$\frac{dy}{dx} - y = (6x-5)e^{x} \quad \text{with} \quad y(1) = 2e \, .$$

(15) 2. (a) Define what it means for $f(x,y)$ to be a homogeneous function of degree k.

(b) Find the general solution of the ODE below

$$(2y^2 - 3x^2)\,dx - xy\,dy = 0 \, .$$

(15) 3. Find the general solution of the ODE given below

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = 6x \cdot y^{1/2} \, .$$

(15) 4. A particle of mass 2 kg falls from rest towards the earth. If the air resistance is λv^2 where $\lambda = \frac{1}{5}\, kg\, m^{-1}$,

(a) find the velocity of the particle after t seconds, and

(b) the limiting velocity of the particle.

(20) 5. (a) Define what is an integrating factor of the non-exact ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \, . \qquad (*)$$

(b) Find the solution of $(2x^2 - y)\,dx + (x^2 y + x)\,dy = 0$.

(20) 6. The population of a colony satisfies the logistic law $dP/dt = (2000P - P^2)/2000$ where t is measured in years.

(a) If $P(0) = 500$, find the population after t years.

(b) How long will it take for the population to reach 1600?

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Solutions to Test #1 — Spring 2006

1. $\frac{dy}{dx} - y = (6x-5)e^{x}$. This is a linear ODE, so integrating factor $= e^{\int P(x)dx} = e^{\int -1 . dx} = e^{-x}$

$\therefore \quad e^{-x}\frac{dy}{dx} - e^{-x}.y = (6x-5).e^{x}.e^{-x}$

$\therefore \quad \frac{d}{dx}(e^{-x}.y) = 6x-5$

$\therefore \quad e^{-x}.y = 3x^2 - 5x + C$

But $y(1) = 2e$, so $e^{-1}.2e = 3.1^2 - 5.1 + C$

$\therefore \quad C = 4$. So $e^{-x}y = 3x^2 - 5x +$

$\therefore \quad y = (3x^2 - 5x + 4)e^{x}$.

2(a) The function $f(x,y)$ is said to be homogeneous of degree k if $f(tx, ty) = t^k.f(x,y)$.

(b) This is a homogeneous ODE. We have

$$(2y^2 - 3x^2)\,dx = xy\,dy$$

$$\therefore \quad \frac{dy}{dx} = \frac{2y^2-3x^2}{xy} = 2\left(\frac{y}{x}\right) - \frac{3}{(y/x)} \qquad (*)$$

Now put $y = xv$. Then $dy/dx = v + x\,dv/dx$ & $v = y/x$

So (*) becomes $v + x\frac{dv}{dx} = 2v - \frac{3}{v}$

$$\therefore \quad x\frac{dv}{dx} = v - \frac{3}{v} = \frac{v^2-3}{v}$$

$$\therefore \quad \frac{v\,dv}{v^2-3} = \frac{dx}{x} \qquad \therefore \quad \frac{2v\,dv}{v^2-3} = \frac{2\,dx}{x}$$

$$\therefore \quad \ln(v^2-3) = 2\ln x + C$$

$$\therefore \quad v^2 - 3 = e^{2\ln x}.e^{C} = Ax^2 \quad \text{where } A = e^{C}$$

$$\therefore \quad v^2 = Ax^2 + 3. \qquad \therefore \quad \frac{y^2}{x^2} = Ax^2 + 3.$$

$$\therefore \quad y^2 = x^2(Ax^2+3)$$

3. This is a Bernoulli ODE. Multiply by $(1-1/2)y^{-1/2}$ $= \frac{1}{2}y^{-1/2}$ on both sides. We get

$$\frac{1}{2} \cdot y^{-1/2}\frac{dy}{dx} + \frac{1}{2} \cdot \frac{2}{x} \cdot y^{-1/2} \cdot y = 6x \cdot \frac{1}{2}y^{-1/2} \cdot y^{1/2}$$

$$\therefore \quad \frac{1}{2}y^{-1/2}\frac{dy}{dx} + \frac{1}{x}y^{1/2} = 3x \qquad (*)$$

Now put $v = y^{1-1/2} = y^{1/2}$. Then $\frac{dv}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$

So $(*)$ becomes $\frac{dv}{dx} + \frac{1}{x}v = 3x$

This is now a linear ODE. So $I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$$\therefore \quad x\frac{dv}{dx} + x \cdot \frac{1}{x} \cdot v = 3x \cdot x^2$$

$$\therefore \quad \frac{d}{dx}(xv) = 3x^2 \qquad \therefore \quad xv = x^3 + C$$

So $\quad x \cdot y^{1/2} = x^3 + C. \quad \therefore \quad y = \left(\frac{x^3+C}{x}\right)^2$

4(a) We have $m\frac{dv}{dt} = mg - \lambda v^2$. So

$$\frac{dv}{dt} = g - \frac{\lambda}{m}v^2 = 10 - \frac{1}{10}v^2 = \frac{100-v^2}{10}.$$

$$\therefore \quad \frac{10\,dv}{10^2-v^2} = dt \qquad \therefore \quad \frac{20\,dv}{(10-v)(10+v)} = 2dt$$

$$\therefore \quad \left(\frac{1}{10+v} + \frac{1}{10-v}\right)dv = 2dt \qquad \therefore \quad \ln(10+v) - \ln(10-v) = 2t + C$$

$$\therefore \quad \ln\left(\frac{10+v}{10-v}\right) = 2t + C \qquad \therefore \quad \frac{10+v}{10-v} = e^c \cdot e^{2t} = A \cdot e^{2t}$$

But $v(0) = 0$, so $\frac{10+0}{10-0} = A \cdot e^{2(0)}$. $\therefore \quad A = 1.$

$$\therefore \quad \frac{10+v}{10-v} = e^{2t}. \qquad \therefore \quad \frac{10-v}{10+v} = e^{-2t}$$

$$\therefore \quad 10 - v = 10 \cdot e^{-2t} + ve^{-2t} \qquad \therefore \quad v(1+e^{-2t}) = 10(1-e^{-2t})$$

$$\therefore \quad v(t) = 10 \cdot \frac{1-e^{-2t}}{1+e^{-2t}}$$

(b) limiting velocity $= \lim_{t\to\infty} v(t) = \lim_{t\to\infty} 10 \cdot \frac{1-e^{-2t}}{1+e^{-2t}}$

$$= 10 \cdot \frac{1-0}{1+0} = 10\ ms^{-1}.$$

5(a) An integrating factor of the ODE (*) is any function $\mu(x,y)$ such that $\mu(x,y)M\,dx + \mu(x,y)N\,dy = 0$ is exact.

(b) This ODE is not exact because $\frac{\partial M}{\partial y} = -1$ & $\frac{\partial N}{\partial x} = 2xy+1$

but $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1-(2xy+1)}{(xy+1)x} = \frac{-2}{x}$.

So an integrating factor is $e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$

So $x^{-2}.(2x^2-y).\,dx + x^{-2}.(x^2y+x)\,dy = 0$ is exact.

$$(2 - yx^{-2})\,dx + (y + x^{-1})\,dy = 0$$

$\therefore \frac{\partial F}{\partial x} = 2 - yx^{-2}$ & $\frac{\partial F}{\partial y} = y + x^{-1}$

$\therefore F = 2x + yx^{-1} + \varphi(y)$

$\therefore \frac{\partial F}{\partial y} = 0 + x^{-1} + \varphi'(y) \Rightarrow \varphi'(y) = y \Rightarrow \varphi(y) = \frac{y^2}{2}$

$\therefore F = 2x + y/x + y^2/2$. $\therefore 2x + \frac{y}{x} + \frac{y^2}{2} = C$.

6(a) We $\frac{dP}{dt} = \frac{(2000-P)P}{2000}$ $\therefore \frac{2000\,dP}{P(2000-P)} = dt$

$\therefore \left(\frac{1}{P} + \frac{1}{2000-P}\right)dP = dt$

$\therefore \ln P - \ln(2000-P) = t + C$

$\therefore \ln\left(\frac{P}{2000-P}\right) = t + C$ $\therefore \frac{P}{2000-P} = e^t.e^C$

$\therefore \frac{P}{2000-P} = A.e^t$ where $A = e^C$

But $P(0) = 500$. So $\frac{500}{2000-500} = A.e^0 \Rightarrow A = \frac{1}{3}$

$\therefore \frac{P}{2000-P} = \frac{e^t}{3}$. $\therefore 3P = 2000e^t - Pe^t$

$\therefore P(3+e^t) = 2000e^t \Rightarrow P = \frac{2000e^t}{3+e^t}$

(b) When $P = 1600$, we have $1600 = \frac{2000e^t}{3+e^t}$

$\therefore 1600 = \frac{2000}{3e^{-t}+1} \Rightarrow 4 = \frac{5}{3e^{-t}+1}$

$\therefore 12e^{-t} + 4 = 5 \Rightarrow 12e^{-t} = 1 \Rightarrow e^t = 12 \Rightarrow t = \ln(12)$.

So it will take $\ln(12)$ years for the population to reach 1600.