MAP 2302 - Differential Equations — Florida Int'l Univ.

Test #2 – Spring 2006 — TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. No calculators or formula sheets are allowed.

(15) 1. Find the solution of each of the following ODE

(a) $y'' - 6y' + 9y = 0$ with $y(0) = 1$ & $y'(0) = 5$

(b) $y'' + 4y = 0$ with $y(0) = 3$ & $y'(0) = -8$.

(20) 2. Find the general solution of each of the following ODEs

(a) $y'' - 3y' + 2y = 4x^2$ (b) $y'' + 2y' - 3y = 12e^x$.

(20) 3. Find y_c and give the minimal form of the y_p that one should try for each of the following ODE

(a) $(D^2 + 2D + 5)^2 y = 3xe^{-x}\sin(2x)$

(b) $(D^2+1)^2(D-2)^3 y = 6\cos^2 x + 2\cos x$

(15) 4.(a) Define what it means for $\{f_1, \ldots, f_n\}$ to be linearly dependent.

(b) Given that $f(x) = x$ is a solution of $(x^2+1)y'' - 4xy' + 4y = 0$, find a second linearly independent solution.

(15) 5(a) Write down what Hooke's law says.

(b) Find a particular solution of the ODE $y'' + y = \sec^2 x$.

(15) 6. A body of mass 4 kg is attached to a Hooke-type spring with spring constant $k = 20\,Nm^{-1}$ on a horizontal frictionless surface. The air resistance to the motion is also λv where $\lambda = 16\,Nsm^{-1}$. If the spring is extended by an amount of 3 m at time $t = 0$ and then let loose, find the position of the body at all subsequent times.

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Solution to Test #2 Spring 2006

1(a) $y'' - 6y' + 9y = 0 \quad \therefore \quad (D^2 - 6D + 9)y = 0$

$\therefore \quad (D-3)^2 y = 0 \Rightarrow (D-3)^2 = 0 \Rightarrow D = 3$ (twice)

$\therefore \quad y = (C_1 + C_2 x)e^{3x}$ is the general solution.

$\therefore \quad y' = [C_2 + 3(C_1 + C_2 x)]e^{3x}$

$y(0) = 1 \Rightarrow \quad C_1 + C_2 \cdot 0 = 1$

$y'(0) = 5 \Rightarrow \quad C_2 + 3(C_1 + C_2 \cdot 0) = 5$

$\therefore \quad C_1 = 0$. So $C_2 = 5 - 3C_1 = 2$

$\therefore \quad y = (1 + 2x)e^{3x}$

(b) $y'' + 4y = 0 \quad \therefore \quad (D^2 + 4)y = 0 \Rightarrow D = \pm 2i$

$\therefore \quad y = C_1 \cos(2x) + C_2 \sin(2x)$ is the general solution

$\therefore \quad y = -2C_1 \sin(2x) + 2C_2 \cos(2x)$

$y(0) = 3 \Rightarrow \quad C_1 \cdot 1 + C_2 \cdot 0 = 3 \Rightarrow C_1 = 3$

$y'(0) = -8 \Rightarrow \quad -2C_1 \cdot 0 + 2C_2 \cdot 1 = -8 \Rightarrow C_2 = -4$

$\therefore \quad y = 3\cos(2x) - 4\sin(2x)$.

2(a) $y'' - 3y' + 2y = 0 \Rightarrow (D^2 - 3D + 2)y = 0$

$\therefore \quad (D-1)(D-2)y = 0 \Rightarrow y_c = C_1 e^x + C_2 e^{2x}$

Try $y_p = Ax^2 + Bx + C$. Then $y_p' = 2Ax + B$

and $y_p'' = 2A$. So $y_p'' - 3y_p' + 2y_p = 4x^2$ becomes

$$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2$$

$\therefore \quad 2A = 4 \quad$ (coeff. of x^2) $\Rightarrow A = 2$

$-6A + 2B = 0 \quad$ (coeff. of x) $\Rightarrow B = 6$

$2A - 3B + 2C = 0 \quad$ (coeff of x^0) $\Rightarrow C = 7$

$\therefore \quad y_p = 2x^2 + 6x + 7$. Hence $y = y_c + y_p$

$= C_1 e^x + C_2 e^{2x} + 2x^2 + 6x + 7$

2(b) $y'' + 2y' - 3y = 0 \Rightarrow (D^2 + 2D - 3)y = 0$

$\Rightarrow (D-1)(D+3)y = 0 \Rightarrow y_c = C_1 e^x + C_2 e^{-3x}$

Try $y_p = Axe^x$ (because 1 is a root of mult. 1)

Then $y_p' = (Ax + A)e^x$ and $y_p'' = (Ax + 2A)e^x$.

So $y_p'' + 2y_p' - 3y_p = 12e^x$ becomes

$$[(Ax + 2A) + 2(Ax + A) - 3(Ax)]e^x = 12e^x$$

$\therefore$ $4A.e^x = 12e^x \Rightarrow A = 3$. So $y_p = 3xe^x$

$\therefore$ $y = y_c + y_p = C_1 e^x + C_2 e^{-3x} + 3xe^x$.

3(a) $(D^2 + 2D + 5)^2 y = 0 \Rightarrow D = (-2 \pm \sqrt{4-20})/2 = -1 \pm 2i$ (twice)

$\therefore$ $y_c = e^{-x}\left[(C_1 + C_2 x)\cos(2x) + (C_3 + C_4 x)\sin(2x)\right]$

The minimal form of y_p that one should try is

$$y_p = e^{-x}.x^2.\left[(A_0 + A_1 x)\cos(2x) + (B_0 + B_1 x)\sin(2x)\right]$$

because $3x$ is a polynomial of degree 1 and $-1 \pm 2i$ is a root of multiplicity 2.

(b) $(D^2+1)^2(D-2)^3 y = 0 \Rightarrow D = 2$ (3 times) or $D = \pm i$ (twice)

$\therefore$ $y_c = (C_1 + C_2 x + C_3 x^2)e^{2x} + (C_4 + C_5 x)\cos x + (C_6 + C_7 x)\sin x$

Now observe that $6\cos^2 x = 6.\frac{1}{2}[1 + \cos(2x)] = 3 + 3\cos(2x)$

So $(D^2+1)^2(D-2)^3 y = 6\cos^2 x + 2\cos x$

$= 3 + 3\cos(2x) + 2\cos x$

Since $\pm i$ is a root of multiplicity 2 we should try $(A_0 \cos x + B_0 \sin x).x^2$ as a part of y_p to cater for the "$2\cos x$". Also since 3 and $3\cos(2x)$ are not involved in y_c, we should add $A_1\cos(2x) + B_1\sin(2x) + A_2$ to our y_p to cater for the "$3\cos(2x)$" and the "3". Hence the minimal form of the y_p that one should try is

$y_p = (A_0\cos x + B_0 \sin x).x^2 + A_1\cos(2x) + B_1\sin(2x) + A_2$.

4(a) The functions $f_1, \dots, f_n$ are linearly dependent if we can find constants $c_1, \dots, c_n$ which are not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) \equiv 0$.

(b) A second linearly indep. solution is given by $y_2 = v \cdot f(x)$ where

$$v = \int \frac{e^{-\int [a_1(x)/a_0(x)]dx}}{[f(x)]^2}\, dx = \int \frac{e^{-\int \frac{-4x}{x^2+1}dx}}{[x]^2} dx$$

$$= \int \frac{e^{2\ln(x^2+1)}}{x^2} dx = \int \frac{e^{\ln(x^2+1)^2}}{x^2} dx = \int \frac{(x^2+1)^2}{x^2} dx$$

$$= \int \frac{x^4+2x^2+1}{x^2} dx = \int \left(x^2+2+x^{-2}\right) dx = \frac{x^3}{3} + 2x - x^{-1}$$

$$\therefore\ y_2(x) = v \cdot f(x) = x\left[\frac{x^3}{3} + 2x - x^{-1}\right] = \tfrac{1}{3}x^4 + 2x^2 - 1.$$

5(a) Hooke's Law: The force with which a Hooke-type spring pulls back is directly proportional to the amount it is extended from its natural length.

(b) We know one $y_p = v_1 y_1 + v_2 y_2$ where y_1 & y_2 are two linearly indep. solutions of the homog. equation and $v_1' = \begin{vmatrix} 0 & y_2 \\ F(x) & y_2' \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ & $v_2' = \begin{vmatrix} y_1 & 0 \\ y_1' & F(x) \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Homog. Eq. is $y'' + y = 0$. So take $y_1 = \cos x$ & $y_2 = \sin x$

$$\therefore\ v_1' = \begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{-\sin x \cdot \sec^2 x}{\cos^2 x + \sin^2 x}$$

$$= -\frac{\sin x}{\cos x} \cdot \sec x = -\tan x \cdot \sec x$$

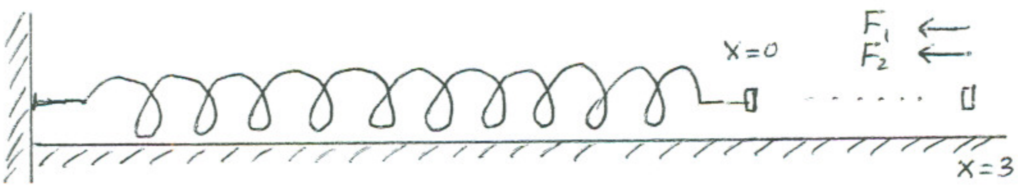
$$\therefore\ v_1 = \int -\tan x \sec x = -\sec x = -1/\cos x.$$

Also
$$v_2' = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{\cos x \cdot \sec^2 x}{\cos^2 x + \sin^2 x}$$

$$= \frac{1}{\sec x} \cdot \sec^2 x = \sec x. \quad \therefore\ v_2 = \int \sec x\, dx = \ln(\sec x + \tan x)$$

$$\therefore\ y_p = \cos x \cdot \frac{-1}{\cos x} + \sin x \cdot \ln(\sec x + \tan x) = -1 + \sin x \ln(\sec x + \tan x).$$

6.



$m\ddot{x} = -F_1 - F_2 = -\lambda v - kx$

$\therefore\ m\ddot{x} + \lambda\dot{x} + kx = 0$

$\therefore\ 4\ddot{x} + 16\dot{x} + 20x = 0 \quad\Rightarrow\quad \ddot{x} + 4\dot{x} + 5x = 0$

$\therefore\ D^2 + 4D + 5 = 0 \Rightarrow D = (-4 \pm \sqrt{16-20})/2 = -2 \pm i$

So $x(t) = e^{-2t}(A\cos t + B\sin t)$

$\therefore\ \dot{x}(t) = -2e^{-2t}(A\cos t + B\sin t) + e^{-2t}(-A\sin t + B\cos t)$

$\quad = e^{-2t}\left[(B-2A)\cos t - (2A+B)\sin t\right]$

$x(0) = 3 \Rightarrow \quad A\cdot 1 + B\cdot 0 = 3 \quad\Rightarrow\quad A = 3$

$\dot{x}(0) = 0 \Rightarrow \quad (B-2A)\cdot 1 \quad (2A+B)\cdot 0 = 0 \quad\Rightarrow\quad B = 6$

$\therefore\ x(t) = 3e^{-2t}(\cos t + 2\sin t)$ END

The question did not ask for a sketch of $x(t)$ but if it did, this is how you would do that.

$3\sqrt{5}$

$x(t) = 3e^{-2t}\cdot\sqrt{5}\left[\cos t\cdot\frac{1}{\sqrt{5}} + \sin t\cdot\frac{2}{\sqrt{5}}\right]$

$\quad = 3\sqrt{5}\,e^{-2t}(\cos t\cos\alpha + \sin t\sin\alpha)$

$\quad = 3\sqrt{5}\,e^{-2t}\cos(t-\alpha)$ where $\tan\alpha = \frac{2}{1} = 2$.