MAP 2302 - Differential Equations — Florida Int'l Univ.

TEST #3 – Spring 2006 — TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE

(15) 1. (a) Find $\mathcal{L}\{t^2\}(s)$ directly from the definition of the Laplace transform.

(b) Define when $x_0$ is a regular singular point of the ODE $y'' + P_1(x)y' + P_2(x)y = 0$.

(15) 2. Find the general solution of the ODE $4x^2y'' - 3y = 6$ by transforming it into a non-homogeneous constant coefficient linear equation in $y$ and $t$.

(20) 3. Solve the following ODEs by using Laplace transforms

(a) $y'(t) - 3y(t) = 4e^{-t}$ with $y(0) = 1$

(b) $y''(t) + 9y(t) = 0$ with $y(0) = 4$ & $y'(0) = -6$.

(20) 4. For each of the following ODEs, find the indicial equation and the form of two lin. indep. series solution about $x_0 = 0$.

(a) $x^2y'' + xy' + (x-2)y = 0$ (b) $x^2y'' + \left(x - \frac{3}{4}\right)y = 0$

(15) 5. Find the first 4 non-zero terms of the power series solution of $y'' + 2xy = 0$ with $y(0) = 3$ & $y'(0) = 2$.

(15) 6. We know that $\mathcal{L}\{\cos(bt)\}(s) = \frac{s}{s^2+b^2}$. Starting with this fact, use any of the Theorems proved in class to find

(a) $\mathcal{L}\{t\cos t\}(s)$ (b) $\mathcal{L}\{\cos^2 t\}(s)$.

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Solutions to Test # 3　　　Spring 2006.

1(a) $\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt = \lim_{R\to\infty} \int_0^R t^2 e^{-st}\,dt$

$= \lim_{R\to\infty}\left[ -\frac{t^2}{s} e^{-st} \Big]_0^R + \frac{1}{s}\int_0^R e^{-st}\cdot 2t\cdot dt \right.$ ← $\boxed{\begin{array}{l} \text{Put } u = t^2 \ \& \ dv = e^{-st}dt \\ du = 2t dt \quad v = -\frac{1}{s}e^{-st} \end{array}}$

$= 0 + \frac{2}{s}\lim_{R\to\infty} \int_0^R e^{-st}\cdot t\,dt$

$\boxed{\begin{array}{l} \text{Put } u = t \ \& \ dv = e^{-st}dt \\ du = dt \ \& \ v = -\frac{1}{s}e^{-st} \end{array}}$

$= \frac{2}{s}\lim_{R\to\infty}\left\{ \underbrace{\left[-\frac{t}{s}e^{-st}\right]_0^R}_{\to 0} + \int_0^R \frac{1}{s}e^{-st}dt \right\}$

$= \frac{2}{s}\cdot \lim_{R\to\infty} \left[\frac{-1}{s^2} e^{-st}\right]_0^R = \frac{2}{s^3}.$

(b) We say that $x_0$ is a singular point of $y'' + P_1(x)y' + P_2(x)y = 0$ if at least one of the two functions $P_1(x)$ & $P_2(x)$ is not analytic at $x_0$. We say that $x_0$ is a regular singular point if $x_0$ is a singular point and both $(x-x_0)P_1(x)$ & $(x-x_0)^2 P_2(x)$ are analytic at $x_0$.

2. $4x^2y'' - 3y = 6$　(*).　Put $x = e^t$. Then $t = \ln x$ and $x^2y'' = D^2y - Dy$ & $xy' = Dy$.

Here $Dy = \frac{d}{dt}(y)$. So (*) becomes

$4(D^2y - Dy) - 3y = 6$　(**)

$\therefore\ (4D^2 - 4D - 3)y = 6 \Rightarrow$

Homog. Eq. $(4D^2 - 4D - 3)y = 0 \Rightarrow 4D^2 - 4D - 3 = 0$

$\Rightarrow (2D+1)(2D-3) \Rightarrow D = -1/2$ or $3/2$.

$\therefore\ y_c(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$

Try $y_p(t) = A$. Then $\dot{y}_p(t) = 0$ & $\ddot{y}_p(t) = 0$. So (**) becomes $0 - 0 - 3A = 6 \Rightarrow A = -2$.

$\therefore\ y = c_1 e^{-t/2} + c_2 e^{3t/2} = c_1 x^{-1/2} + c_2 x^{3/2} - 2.$

3 (a) $y'(t) - 3y(t) = 4e^{-t}$ $\quad y(0) = 1$

$\therefore \mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 4\mathcal{L}\{e^{-t}\}$

$\therefore s\mathcal{L}\{y\} - y(0) - 3\mathcal{L}\{y\} = \frac{4}{s+1}$

$\therefore (s-3)\mathcal{L}\{y\} = \frac{4}{s+1} + y(0) = \frac{4}{s+1} + 1 = \frac{s+5}{s+1}$

$\therefore \mathcal{L}\{y\} = \frac{s+5}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$

$\therefore A(s+1) + B(s-3) = s+5$

Putting $s=3$, gives $A(3+1) = 3+5 \Rightarrow A = 2$

Putting $s=-1$, gives $B(-1-3) = -1+5 \Rightarrow B = -1$

$\therefore \mathcal{L}\{y\} = \frac{2}{s-3} - \frac{1}{s+1}$

$\therefore y(t) = 2e^{3t} - e^{-t}$

(b) $y''(t) + 9y(t) = 0$ $\quad y(0) = 4, \quad y'(0) = -6$

$\therefore \mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = 0$

$\therefore s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 9\mathcal{L}\{y\} = 0$

$\therefore (s^2+9)\mathcal{L}\{y\} = sy(0) + y'(0) = 4s - 6$

$\therefore \mathcal{L}\{y\} = \frac{4s-6}{s^2+3^2} = 4 \cdot \frac{s}{s^2+3^2} - 2 \cdot \frac{3}{s^2+3^2}$

$\therefore y(t) = 4\cos(3t) - 2 \cdot \sin(3t)$.

4 (a) $x^2y'' + xy' + (x-2)y = 0$. The associated Cauchy-Euler equation is $x^2y'' + xy' - 2y = 0$. So the indicial equation is $r(r-1) + r - 2 = 0$

$\therefore r^2 - 2 = 0 \Rightarrow r = \pm\sqrt{2}$. $\therefore r_1 = \sqrt{2}, r_2 = -\sqrt{2}$

Since $r_1 - r_2 = 2\sqrt{2}$ is not an integer, we know that there are two linearly indep. solutions of the form

$y_1(x) = x^{\sqrt{2}} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$ with $a_0 \neq 0$

$y_2(x) = x^{-\sqrt{2}} \sum_{n=0}^{\infty} b_n \cdot x^n$ with $b_0 \neq 0$.

4 (b) $x^2 y'' + (x - 3/4) y = 0$. The associated Cauchy-Euler equation is $x^2 y'' - 3/4\, y = 0$. So the indicial equation will be $r(r-1) - 3/4 = 0$

$\therefore\ r^2 - r - 3/4 = 0$. $(r + 1/2)(r - 3/2) = 0$

$\therefore\ r_1 = 3/2$ and $r_2 = -1/2$. Since $r_1 - r_2 = 2$ is a positive integer, we know that two linearly indep. solutions will be of the form

$$y_1(x) = x^{3/2} \cdot \sum_{n=0}^{\infty} a_n x^n \quad \text{with } a_0 \neq 0$$

$$\& \quad y_2(x) = A \cdot y_1(x) \ln x + x^{-1/2} \cdot \sum_{n=0}^{\infty} b_n x^n \quad \text{with } b_0 \neq 0$$

Here $A$ may or may not be zero.

5. $y'' + 2xy = 0$ : $(*)$. $y(0) = 3$ & $y'(0) = 2$

Let $y = \sum_{n=0}^{\infty} a_n . x^n$. Then

$$y' = \sum_{n=0}^{\infty} n a_n . x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

So $(*)$ becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} . x^n + 2x . \sum_{n=0}^{\infty} a_n . x^n = 0$$

$$\therefore \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2 . a_{n-1} . x^n = 0$$

$$\therefore 2a_2 + \sum_{n=1}^{\infty} \{(n+2)(n+1) a_{n+2} + 2a_{n-1}\} x^n = 0$$

$\therefore\ 2a_2 = 0$ & $(n+2)(n+1) a_{n+2} + 2a_{n-1} = 0$ for $n \geq 1$

$\therefore\ a_2 = 0$ & $a_{n+2} = \dfrac{-2 a_{n-1}}{(n+1)(n+2)}$ for $n \geq 1$.

So $y(0) = \sum_{n=0}^{\infty} a_0 . 0^n = a_0 \Rightarrow a_0 = 3$ bec. $0^0 = 1$

$y'(0) = \sum_{n=0}^{\infty} (n+1) . a_{n+1} 0^n = a_1 \rightarrow a_1 = 2$ bec. $0^0 = 1$

$a_3 = -2a_0/(3)(2) = -1$, $a_4 = -2a_1/4(3) = -1/3$

$\therefore\ y(x) = 3 + 2x + 0 . x^2 - x^3 - x^4/3 + \cdots$

6. (a) We know that $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$. So

$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\left[\mathcal{L}\{\cos t\}\right] = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -\frac{1.(s^2+1) - s.(2s)}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

(b) $$\mathcal{L}\{\cos^2 t\} = \mathcal{L}\left\{\tfrac{1}{2} + \tfrac{1}{2}\cos(2t)\right\}$$

$$= \frac{1}{2}\,\mathcal{L}\{1 + \cos 2t\}$$

$$= \frac{1}{2}\cdot\left\{\frac{1}{s} + \frac{s}{s^2+4}\right\} = \frac{1}{2}\,\frac{s^2+4+s^2}{s(s^2+4)}$$

$$= \frac{1}{2}\,\frac{2s^2+4}{s(s^2+4)} = \frac{s^2+2}{s(s^2+4)}.$$