MAP 2302 - Differential Equations   Florida Int'l Univ.

TEST #1 - Spring 2013   TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. No calculators or formula sheets are allowed.

(15) 1(a) Specify exactly when the ODE $dy/dx = f(x,y)$ is homogeneous.

(b) Find the general solution of the ODE

$$(2y^2 + x^2)\,dx - xy\,dy = 0 .$$

(15) 2. Find the solution of the ODE

$$dy/dx - 2.y = 6e^{-x} \qquad \text{with } y(0) = 3 .$$

(15) 3. Show that the ODE $(3x^2 + y^{-1}\sin x)dx + (3\cos y + y^{-2}\cos x)dy = 0$ is exact. Then find the general solution of this ODE.

(15) 4. Find the general solution of the ODE

$$dy/dx - y/(2x) = -3x.y^3 .$$

(20) 5. The population of an island colony of a certain kind of animal satisfy the logistic law $\dfrac{dP}{dt} = -\dfrac{P(P-3000)}{3000}$, where t is measured in years.

(a) If $P(0) = 4000$, find the population after t years.

(b) How long will it take the population to decrease to 3,200 ?

(20) 6. A particle of mass 3kg falls from rest towards the earth. If the air resistance is λv where $\lambda = (3/2)\,kg.s^{-1}$ and $g = 10\,ms^{-2}$,

(a) find the velocity of the particle after t seconds.

(b) find the distance the particle has travelled after t seconds.

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Solutions to Test #1 — Spring 2013

1(a) The ODE $dy/dx = f(x,y)$ is homogeneous if we can find a function g such that $f(x,y) = g(y/x)$.

(b) We have $(2y^2+x^2)\,dx = xy\,dy$

So $dy/dx = (2y^2+x^2)/(xy) = 2(y/x) + (x/y)$

$\therefore$ this is a homogeneous ODE. Now put $y = xv$.

Then $dy/dx = v + x\,dv/dx$ and $v = y/x$. Hence

$$v + x\frac{dv}{dx} = 2v + \frac{1}{v}$$

$\therefore \quad x.\,dv/dx = v + (1/v) = (v^2+1)/v$

$\therefore \quad v\,dv/(v^2+1) = dx/x$.

$\therefore \quad (2v/v^2+1)\,dv = 2\,dx/x$

$\therefore \quad \ln(v^2+1) = 2\ln(x) + C$

$\therefore \quad v^2+1 = e^{2\ln x}.e^C = e^{\ln x^2}.e^C$

$\therefore \quad v^2+1 = Ax^2$; where $A = e^C$

$\therefore \quad v^2 = Ax^2-1$ and so $y^2/x^2 = Ax^2-1$

$\therefore \quad y^2 = x^2(Ax^2-1)$.

2. $dy/dx - 2y = 6.e^{-x}$. This is a linear first order ODE.

So Integrating factor $= e^{\int -2dx} = e^{-2x}$.

$\therefore \quad e^{-2x}.(dy/dx) - y.e^{-2x} = 6.e^{-x}.e^{-2x}$

$\therefore \quad \frac{d}{dx}(ye^{-2x}) = 6e^{-3x}$. So $ye^{-2x} = -2e^{-3x} + C$

$\therefore \quad y = -2e^{-x} + Ce^{2x}$.

But $y(0) = 3$, so $3 = -2.e^0 + C.e^0$

$\therefore \quad C = 3+2 = 5$. Hence

$$y = 5e^{2x} - 2e^{-x}.$$

3 (a) The ODE $Mdx + Ndy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Here $M = 3x^2 + y^{-1}\sin x$, so $\frac{\partial M}{\partial y} = 0 - y^{-2}\sin x$.

And $N = 3\cos y + y^{-2}\cos x$, so $\frac{\partial N}{\partial x} = 0 + y^{-2}(-\sin x)$.

$\therefore$ $\partial M/\partial y = \partial N/\partial x$, so the ODE is exact.

(b) Hence we can find an F such that $dF = Mdx + Ndy$.

Now $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$, so

$$\partial F/\partial x = 3x^2 + y^{-1}\sin x \quad \& \quad \partial F/\partial y = 3\cos y + y^{-2}\cos x$$

$\therefore$ $F = \int(3x^2 + y^{-1}\sin x)\,\partial x = x^3 - y^{-1}\cos x + \varphi(y)$

$\therefore$ $\frac{\partial F}{\partial y} = 0 - (-y^{-2}).\cos x + \varphi'(y)$. But

$\partial F/\partial y = y^{-2}\cos x + 3\cos y$

$\therefore$ $\varphi'(y) = 3\cos(y)$. $\therefore$ $\varphi(y) = 3\sin(y) + C_1$

Hence $F = x^3 - y^{-1}\cos x + \varphi(y)$

$= x^3 - y^{-1}\cos x + 3\sin y + C_1$

But $dF = Mdx + Ndy = 0$, so $F = C_2$

Hence $C_2 = x^3 - y^{-1}\cos x + 3\sin y + C_1$

$\therefore$ $x^3 - y^{-1}\cos x + 3\sin y = C$ where $C = C_2 - C_1$.

4. $dy/dx - y/(2x) = -3x.y^3$. This is a Bernoulli ODE with index $n = 3$. So put $v = y^{1-n} = y^{-2}$ and multiply both sides of the ODE by $(1-n)y^{-n} = -2y^{-3}$ to get:

$$-2y^{-3}.\frac{dy}{dx} - \frac{y}{2x}.(-2y^{-3}) = (-3x).y^3.(-2y^{-3})$$

$\therefore$ $-2y^{-3}\frac{dy}{dx} + \frac{y^{-2}}{x} = 6x$

$\therefore$ $dv/dx + \frac{1}{x}v = 6x$

This is a linear ODE in v. So I.F. $= e^{\int\frac{1}{x}dx} = e^{\ln x} = x$.

$\therefore$ $x\,(dv/dx) + x.(\frac{1}{x}v) = 6x^2$

$\therefore$ $\frac{d}{dx}(xv) = 6x^2 \Rightarrow xv = 2x^3 + C$

$\therefore$ $x.y^{-2} = 2x^3 + C \Rightarrow y^{-2} = (2x^3+C)/x$

$\therefore$ $y^2 = \dfrac{x}{C + 2x^3}$.

5 (a) We have $dP/dt = -P(P-3000)/3000$. Hence

$$\frac{-3000}{P(P-3000)}\,dP = dt \quad \therefore \left(\frac{1}{P} - \frac{1}{P-3000}\right)dP = dt$$

$\therefore \ln P - \ln(P-3000) = t + C \quad \therefore \ln\left(\frac{P}{P-3000}\right) = t+C$

$\therefore P/(P-3000) = e^t . e^C = A.e^t$. But $P(0) = 4000$,

so $4000/(4000-3000) = A.e^0 \Rightarrow A = 4$

$\therefore P/(P-3000) = 4e^t \Rightarrow (P-3000)/P = \frac{1}{4}e^{-t} \quad \dots (*)$

$\therefore 1 - 3000/P = \frac{1}{4}e^{-t} \Rightarrow 3000/P = 1 - \frac{e^{-t}}{4}$

$$\therefore P = \frac{3000}{1-\frac{1}{4}e^{-t}} = \frac{12000}{4-e^{-t}}$$

(b) When $P = 3,200$ we have $\frac{3200-3000}{3200} = \frac{1}{4}e^{-t}$ from $(*)$

$\therefore \frac{200}{3200} = \frac{1}{4}e^{-t} \Rightarrow \frac{800}{3200} = e^{-t} \Rightarrow e^t = 4 \Rightarrow t = \ln 4$.

So population will decrease to 3,200 in $\ln(4)$ years.

6 (a) We have $m\,dv/dt = mg - \lambda v$, $v(0) = 0$ & $x(0) = 0$

where $x(t)$ = distance measured downwards from starting pos.

$\therefore 3\frac{dv}{dt} = 3(10) - \frac{3}{2}v$

$\therefore dv/dt = 10 - v/2 = \frac{1}{2}.(20-v)$

$\therefore \frac{dv}{20-v} = \frac{1}{2}dt \quad \therefore -\ln(20-v) = \frac{t}{2} + C_1$

$\therefore \ln(20-v) = -t/2 - C_1 \Rightarrow 20 - v = e^{-t/2}.e^{-C_1} = Ae^{-t/2}$

But $v(0) = 0$, so $20 - 0 = A.e^0 \Rightarrow A = 20$

$\therefore 20 - v = 20e^{-t/2} \Rightarrow v = 20 - 20e^{-t/2}$

(b) $\therefore dx/dt = 20 - 20e^{-t/2}$

$\therefore dx = (20 - 20e^{-t/2})dt$

$\therefore x(t) = 20t - 20.\left(\frac{1}{-1/2}\right).e^{-t/2} + C_2$

$= 20t + 40e^{-t/2} + C_2$

But $x(0) = 0$. So $0 = 2(0) + 40.e^0 + C_2$

$\therefore C_2 = -40$. $\therefore x(t) = 20t + 40e^{-t/2} - 40$.