MAP 2302 – Differential Equations — Florida Int'l Univ.

TEST #2 – Spring 2013 — TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. NO CALCULATORS OR CELLPHONES ARE ALLOWED.

(20) 1. Find the solution of each of the following homogeneous ODEs

(a) $y'' - y' - 6y = 0$ with $y(0) = 3$ & $y'(0) = 4$.

(b) $y'' + 4y' + 4y = 0$ with $y(0) = -2$ & $y'(0) = 5$.

(15) 2.(a) Define what it means for $\{f_1, f_2, \ldots, f_n\}$ to be linearly independent.

(b) Find the general solution of $y'' + y' - 6y = -16x.e^x$.

(15) 3. Find the general solution of the non-homogeneous ODE

$$y'' + 2y' + 5y = 10\sin x.$$

(20) 4. Find y_c and give the minimal form of the y_p that one should try for each of the following ODEs

(a) $D(D^2 - 2D + 2)^2 y = 3x.e^x.\sin x$.

(b) $(D^2 - 1)(D^2 + 4)y = e^x + 2\sin^2 x$.

(15) 5. Find a particular solution of the ODE, $y'' + y = \tan x$.

(15) 6. A body of mass 2kg is attached to a Hooke-type spring and suspended from the ceiling. The natural length L is 10m, the spring constant k is $4\,Nm^{-1}$, and the air resistance is λv where $\lambda = 4\,Ns\cdot m^{-1}$. If the spring is stretched by an amount of 3m and the body is released from rest at time $t=0$, find the amount by which it will be extended at all subsequent times. [Use $g = 10\,ms^{-2}$.]

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Solutions to Test # 2 — Spring 2013

1(a) $y'' - y' - 6y = 0$. $\therefore (D^2 - D - 6)y = 0$ $\therefore (D+2)(D-3)y = 0$

$\therefore D = -2$ or 3. So $y = Ae^{-2x} + Be^{3x}$. $\therefore y' = -2Ae^{-2x} + 3Be^{3x}$

$y(0) = 3 \Rightarrow 3 = Ae^0 + Be^0 \Rightarrow A + B = 3 \Rightarrow B = 3 - A$

$y'(0) = 4 \Rightarrow 4 = -2Ae^0 + 3Be^0 \Rightarrow -2A + 3B = 4$

$\therefore -2A + 3(3-A) = 4 \Rightarrow -5A = -5 \Rightarrow A = 1$. $\therefore B = 3 - 1 = 2$

So $y = e^{-2x} + 2e^{3x}$.

(b) $y'' + 4y' + 4y = 0$ $\therefore (D^2 + 4D + 4)y = 0$ $\therefore (D+2)^2 y = 0$.

$\therefore D = -2$ (twice). So $y = (A + Bx)e^{-2x}$. Hence

$y' = (0+B)e^{-2x} + (-2)(A+Bx)e^{-2x} = \{(B-2A) - 2Bx\}e^{-2x}$.

$y(0) = -2 \Rightarrow -2 = (A + B.0)e^0 \Rightarrow A = -2$.

$y'(0) = 5 \Rightarrow 5 = \{(B-2A) - 2B.0\}e^0 \Rightarrow B = 2A + 5 = 1$

$\therefore y = (Bx + A)e^{-2x} = (x-2)e^{-2x}$.

2(a) The functions $f_1, f_2, \ldots, f_n$ are linearly independent if

$c_1 f_1(x) + c_2 f(x) + \cdots + c_n f_n(x) \equiv 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$.

(b) The homog. eq. is $y'' + y' - 6y = 0$. $\therefore (D^2 + D - 6)y = 0$.

$\therefore (D-2)(D+3)y = 0$. $\therefore y_c = c_1 . e^{2x} + c_2 . e^{-3x}$

The non-homog. eq. is $y'' + y' - 6y = -16xe^x$ ... (**)

Try $y_p = (Ax + B).e^x$. Then

$y_p' = A.e^x + (Ax+B)e^x = (Ax + A + B)e^x$

and $y_p'' = A.e^x + (Ax + A + B)e^x = (Ax + 2A + B)e^x$.

So (**) becomes

$(Ax + 2A + B).e^x + (Ax + A + B)e^x - 6(Ax + B)e^x = -16x.e^x$

$\therefore \{-4Ax + (3A - 4B)\}.e^x = -16x.e^x$

$\therefore -4A = -16 \Rightarrow A = 4$ & $3A - 4B = 0 \Rightarrow B = \frac{3A}{4} = 3$

$\therefore y_p = (4x+3)e^x$. $\therefore y = y_c + y_p = c_1 e^{2x} + c_2 e^{-3x} + (4x+3)e^x$.

3. The homog. eq. is $y'' + 2y' + 5y = 0$. So $(D^2+2D+5)y = 0$

$\therefore\ D = [-2 \pm \sqrt{4-4(5)}]/2 = (-2 \pm \sqrt{-16})/2 = -1 \pm 2i$

$\therefore\ y_c = C_1 e^{-x} \cos(2x) + C_2 . e^{-x} \sin(2x)$.

The non-homog. eq. is $y'' + 2y' + 5y = 10 \sin x \ldots (**)$. Try

$y_p = A \cos x + B \sin x$. Then

$y_p' = -A \sin x + B \cos x$ & $y_p'' = -A \cos x - B \sin x$

So $(**)$ becomes

$(-A \cos x - B \sin x) + 2(B \cos x - A \sin x) + 5(A \cos x + B \sin x) = 10 \sin x$

$\therefore\ (-A + 2B + 5A)\cos x + (-B - 2A + 5B) \sin x = 10 \sin x$

$\therefore\ -A + 5A + 2B = 0 \Rightarrow B = -2A$ & $-2A - B + 5B = 10$ which implies $-2A + 4B = 10 \Rightarrow -2A - 8A = 10 \Rightarrow A = -1$, so $B = 2$

$\therefore\ y_p = -\cos x + 2 \sin x$.

$\therefore\ y = y_c + y_p = C_1 e^{-x} \cos(2x) + C_2 e^{-x} \sin x + 2 \sin x - \cos x$.

4(a) $D(D^2-2D+2)^2 y = 3xe^x \sin x$. $D(D^2-2D+2)^2 = 0 \Rightarrow D = 0$, or $D = [-(-2) \pm \sqrt{4-4(2)}]/2 = (2 \pm 2i)/2 = 1 \pm i$ (twice).

$\therefore\ y_c = C_1 + (C_2 + C_3 x) e^x \cos x + (C_4 + C_5 x) e^x \sin x$

Since $3x$ is a polynomial of degree 1 and $1 \pm i$ are roots of the aux. eq. of multiplicity 2, the minimal y_p is given by

$y_p = [(A_1 + A_2 x) e^x \cos x + (B_1 + B_2 x) e^x \sin x] . x^2$

(b) $(D^2-1)(D^2+4) y = e^x + 2\sin^2 x = e^x + 2(1-\cos 2x)/2 = e^x + 1 - \cos(2x)$.

$(D^2-1)(D^2+4) = 0 \Rightarrow (D-1)(D+1)(D-2i)(D+2i) = 0 \Rightarrow D = 1$, $D = -1$, or $D = \pm 2i$. Hence

$y_c = C_1 e^x + C_2 e^{-x} + C_3 \cos(2x) + C_4 \sin(2x)$.

Since $D = 1$ & $D = \pm 2i$ are roots of the aux. equation of degree 1, and $D = 0$ is not a root of the aux. equation the minimal y_p that one should try is

$y_p = A_0 + A_1 x . e^x + A_3 . x . \cos(2x) + A_4 . x . \sin(2x)$

5. One $y_p = v_1 y_1 + v_2 y_2$ where y_1 & y_2 are two linearly independent solutions of the homogenous equation and

$$v_1' = \begin{vmatrix} 0 & y_2 \\ F(x) & y_2' \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \& \quad v_2' = \begin{vmatrix} y_1 & 0 \\ y_1' & F(x) \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The homog. eq. is $y'' + y = 0$. So take $y_1 = \cos x$ & $y_2 = \sin x$.

$$\therefore\ v_1' = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{-\sin x . \tan x}{\cos^2 x + \sin^2 x} = -\frac{\sin^2 x}{\cos x}$$

$$= -(1-\cos^2 x)/\cos x = \cos x - 1/\cos x = \cos x - \sec x .$$

$$\therefore\ v_1 = \int (\cos x - \sec x)\,dx = \sin x - \ln(\sec x + \tan x)$$

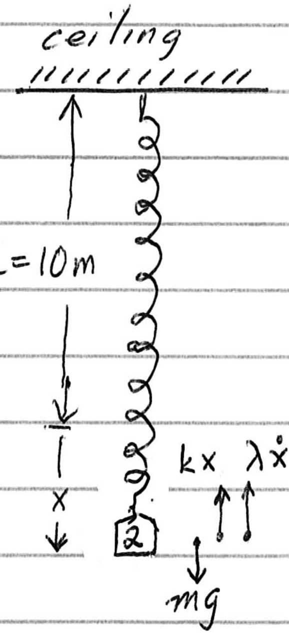
Also $$v_2' = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} \Big/ \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{\cos x . \tan x}{\cos^2 x + \sin^2 x} = \frac{\cos x}{1} . \frac{\sin x}{\cos x}$$

$$\therefore\ v_2 = \int \sin x \, dx = -\cos x$$

$$\therefore\ y_p = v_1 y_1 + v_2 y_2 = [\sin x - \ln(\sec x + \tan x)]\cos x + [-\cos x]\sin x$$

$$= -(\cos x) . \ln(\sec x + \tan x).$$

6. Let $x(t)$ = amount the spring is extended at time t.



Then $m\ddot{x} = mg - kx - \lambda\dot{x}$, so $\ddot{x} + \frac{\lambda}{m}\dot{x} + \frac{k}{m}x = g$.

$\therefore\ \ddot{x} + 2\dot{x} + 2x = 10 \Rightarrow (D^2 + 2D + 2)x = 10$.

$D^2 + 2D + 2 = 0 \Rightarrow D = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$.

$\therefore\ x_c(t) = e^{-t}(A\cos t + B\sin t)$.

Try $x_p(t) = C$. Then $\dot{x}_p(t) = 0$ & $\ddot{x}_p(t) = 0$.

So $\ddot{x} + 2\dot{x} + 2x = 10$ becomes $0 + 0 + 2C = 10 \Rightarrow C = 5$.

$\therefore\ x(t) = x_c(t) + x_p(t) = 5 + e^{-t}(A\cos t + B\sin t)$

$\therefore\ \dot{x}(t) = 0 - e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t)$

$= e^{-t}[(B-A)\cos t - (A+B)\sin t]$.

Now $x(0) = 3$ and $\dot{x}(0) = 0$ because the spring was stretched by 3m & the body was released from rest at time $t = 0$.

So $3 = 5 + e^{-0}(A\cos(0) + B\sin(0)) \Rightarrow A = -2$

and $0 = e^{-0}[(B-A)\cos(0) - (A+B)\sin(0)] \Rightarrow B = A = -2$

$\therefore\ x(t) = 5 - 2e^{-t}(\cos t + \sin t)$

$= 5 - 2\sqrt{2}e^{-t}\left(\cos t . \frac{1}{\sqrt{2}} + \sin t . \frac{1}{\sqrt{2}}\right) = 5 - 2\sqrt{2}e^{-t} . \cos\left(t - \frac{\pi}{4}\right)$.