MAP 2302 - Differential Equations Florida Int'l Univ.

Test #3 - Spring 2013 TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. Begin each question on a sep. page.

(15) 1. Starting with $\mathcal{L}\{\cos(bt)\}(s) = \frac{s}{s^2+b^2}$, use the properties of the Laplace Transform to find
(a) $\mathcal{L}\{\sin^2(t)\}(s)$ (b) $\mathcal{L}\{t.\sin^2(t)\}(s)$

(15) 2. Find the general solution of the ODE $x^2y'' - 2y = 4(\ln x)$ by transforming it first into a linear non-homogeneous constant coefficient ODE in y and t.

(15) 3(a) Define what it means for x_0 to be a regular singular point of the ODE $y'' + P_1(x)y' + P_2(x)y = 0$.
(b) Find the general solution of the ODE $x^2y'' - 3xy' + 5y = 0$.

(20) 4. Solve the following IVPs by using the Laplace transform.
(a) $y''(t) + 4.y(t) = 0$ with $y(0) = 3$ & $y'(0) = -2$.
(b) $y'(t) + 2.y(t) = -3e^t$ with $y(0) = 1$.

(20) 5. For each of the following ODEs, find the indicial equation and the form of two linearly independent Frobenius series solution about $x_0 = 0$.
(a) $4x^2y'' + 8x.y' + (1-x)y = 0$ (b) $x^2y'' + x^3.y' + (x-3/4)y = 0$.

(15) 6. Find the first 5 non-zero terms of the power series solution of the ODE $y'' - x.y' + y = 0$ with the initial conditions $y(0) = 3$ and $y'(0) = 2$.

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Solutions to Test #3 — Spring 2013

1(a) $\mathcal{L}\{\sin^2(t)\}(s) = \mathcal{L}\{\frac{1}{2}[1-\cos(2t)]\} = \frac{1}{2}[\mathcal{L}\{1\} - \mathcal{L}\{\cos(2t)\}$

$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \frac{1}{2}\cdot\frac{4}{s(s^2+4)} = \frac{2}{s(s^2+4)}$

(b) $\mathcal{L}\{t.\sin^2 t\}(s) = -\frac{d}{ds}[\mathcal{L}\{\sin^2(t)\}(s)] = -\frac{d}{ds}\left(\frac{2}{s^3+4s}\right)$

$= --\frac{2(s^3+4s)'}{(s^3+4s)^2} = \frac{2(3s^2+4)}{s^2(s^2+4)^2}$.

2. Put $x=e^t$. Then $t=\ln(x)$, $xy' = D\dot{y}$ & $x^2y'' = D(D-1)y$ where $D = d/dt$. So the homog. equation $x^2y'' - 2y = 0$ becomes $[D(D-1) - 2]y = 0$

$\therefore D^2 - D - 2 = 0 \Rightarrow (D+1)(D-2) = 0 \Rightarrow D = -1$ or 2.

So $y_c = c_1e^{-t} + c_2e^{2t} = c_1x^{-1} + c_2x^2$. Now the non-homog. equation is $(D^2-D-2)y = 4t$.

So try $y_p = At+B$. Then $\dot{y} = A$ & $\ddot{y} = 0$.

So $\ddot{y} - \dot{y} - 2y = 4t$ becomes

$0 - A - 2(At+B) = 4t$. $\therefore -2A = 4$ & $-A-2B = 0$

So $A = -2$ & $B = -\frac{A}{2} = 1$. $\therefore y_p = -2t+1 = -2\ln(x)+1$

$\therefore y = y_c + y_p = c_1x^{-1} + c_2x^2 - 2\ln(x) + 1$.

3(a) x_0 is a singular point of the ODE $y'' + P_1(x)y' + P_2(x)y = 0$ if at least one of the functions $P_1(x)$ & $P_2(x)$ is not analytic at x_0. x_0 is a regular singular point of this same ODE if x_0 is a singular point of the ODE and both $(x-x_0)P_1(x)$ & $(x-x_0)^2P_2(x)$ are analytic at x_0.

(b) Put $x=e^t$. Then $t=\ln(x)$, $xy' = Dy$ & $x^2y'' = D(D-1)y$ where $D = \frac{d}{dt}$. So $x^2y'' - 3xy' + 5y = 0$ becomes $[D(D-1) - 3D + 5]y = 0$, $\therefore (D^2 - 4D + 5)y = 0$.

3 (b) $\therefore D = [-(-4) \pm \sqrt{16-20}]/2 = (4 \pm 2i)/2 = 2 \pm i$

$\therefore y(t) = e^{2t}.(c_1 \cos t + c_2 \sin t)$

$\therefore y(x) = x^2 [c_1 . \cos(\ln x) + c_2 \sin(\ln x)]$.

4 (a) $y''(t) + 4y(t) = 0$ and $y(0) = 3$ & $y'(0) = -2$

$\therefore \mathcal{L}\{y''\} + \mathcal{L}\{4y\} = 0$

$\therefore s^2 \mathcal{L}\{y\} - s.y(0) - y'(0) + 4\mathcal{L}\{y\} = 0$

$\therefore (s^2+4)\mathcal{L}\{y\} = s.y(0) + y'(0) = 3s - 2$

$\therefore \mathcal{L}\{y\} = \dfrac{3s-2}{s^2+4} = 3 . \dfrac{s}{s^2+2^2} - 1 . \dfrac{2}{s^2+2^2}$

$\therefore y(t) = 3\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2^2}\right\} - 1 . \mathcal{L}^{-1}\left\{\dfrac{2}{s^2+2^2}\right\} = 3\cos(2t) - \sin(2t)$.

(b) $y'(t) + 2y(t) = -3e^t$ and $y(0) = 1$

$\therefore \mathcal{L}\{y'\} + \mathcal{L}\{2y\} = \mathcal{L}\{-3e^t\}$

$\therefore s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\} = -3\mathcal{L}\{e^t\}$

$\therefore (s+2)\mathcal{L}\{y\} = y(0) - 3/(s-1) = 1 - 3/(s-1) = \dfrac{s-4}{s-1}$

$\therefore \mathcal{L}\{y\} = \dfrac{s-4}{(s+2)(s-1)} = \dfrac{A}{s+2} + \dfrac{B}{s-1}$

$\therefore s - 4 = A(s-1) + B(s+2)$

Putting $s = -2$ gives us $-2-4 = A(-2-1) \Rightarrow A = 2$

Putting $s = 1$ gives us $1 - 4 = B(1+4) \Rightarrow B = -1$

$\therefore \mathcal{L}\{y\} = \dfrac{2}{s+2} - \dfrac{1}{s-1}$ $\therefore y(t) = 2e^{-2t} - e^t$.

5 (a) The ODE is $4x^2y'' + 8xy' + (1-x)y = 0$. So the associated Cauchy-Euler ODE is $4x^2y'' + 8xy' + y = 0$.

The auxiliary eq. of this is $4D(D-1) + 8D + 1 = 0$

So the indicial equation is $4r(r-1) + 8r + 1 = 0$

$\therefore 4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2 = 1 \Rightarrow r = -1/2$ (twice)

$\therefore y_1(x) = x^{-1/2} \sum_{n=0}^{\infty} a_n x^n$ & $y_2(x) = y_1(x)\ln(x) + x^{-1/2}\sum_{n=1}^{\infty} b_n x^n$

where $a_0 = 1$ & where it is possible for all the b_n's to be 0.

5(b) The ODE is $x^2y'' + x^3 y' + (x-3/4)y = 0$. So the associated Cauchy-Euler ODE is $x^2y'' - (3/4)y = 0$. The auxiliary eq. of this is $D(D-1) - 3/4 = 0$. So the indicial eq. is $r(r-1) - 3/4 = 0$. $\therefore r^2 - r - 3/4 = 0$.

$\therefore r = [-(-1) \pm \sqrt{1+3}]/2 = (1 \pm 2)/2 = 3/2$ or $-1/2$.

Since the roots differ by an integer, we get two linearly independent solutions of the form

$y_1(x) = x^{3/2} \sum_{n=0}^{\infty} a_n x^n$ & $y_2(x) = A.y_1(x)\ln(x) + x^{-1/2}\sum_{n=0}^{\infty} b_n x^n$

with $a_0 = 1 = b_0$ & where A may or may not be 0.

6. Let $y = \sum_{n=0}^{\infty} a_n . x^n$. Then $y' = \sum_{n=1}^{\infty} n.a_n.x^{n-1}$ and

$y'' = \sum_{n=2}^{\infty} n(n-1).a_n.x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}.x^n$. So

$y'' - xy' + y = 0$ becomes

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - x.\sum_{n=1}^{\infty} n.a_n.x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\therefore [(0+1)(0+2)a_2 + a_0].x^0 + \sum_{n=1}^{\infty}[(n+1)(n+2)a_{n+2} - (n-1)a_n]x^n = 0$$

$\therefore 2a_2 + a_0 = 0$ & $(n+1)(n+2)a_{n+2} - (n-1)a_n = 0$.

Now $y(0) = 3 \Rightarrow \sum_{n=0}^{\infty} a_n.0^n = 3 \Rightarrow a_0 = 3$. and

$y'(0) = 2 \Rightarrow \sum_{n=1}^{\infty} a_n.0^{n-1} = 3 \Rightarrow a_1 = 2$ bec. $0^0 = 1$.

Hence $a_2 = -a_0/2 = -3/2$.

Also $a_{n+2} = [(n-1)/(n+1)(n+2)]a_n$ for $n \geq 1$.

$\therefore a_3 = a_{1+2} = [0/(1+1)(1+2)]a_1 = 0$,

$a_4 = a_{2+2} = [1/(2+1)(2+2)]a_2 = (1/12).\frac{-3}{2} = \frac{-1}{8}$,

$a_5 = a_{3+2} = [2/(3+1)(3+2)]a_3 = 0$,

and $a_6 = a_{4+2} = [3/(4+1)(4+2)]a_4 = \frac{3}{30}.\frac{-1}{8} = \frac{-1}{80}$.

So the first 5 non-zero terms are given by

$y(x) = 3 + 2x - \frac{3}{2}x^2 + 0.x^3 - \frac{1}{8}x^4 + 0.x^5 - \frac{1}{80}x^6 + \ldots$