First solution – doing everything from scratch.

4.1 (d) #7 $(x^2 - 2x + 2)y'' - x^2 y' + xy = 0$ (*), $f(x) = x$ is a sol.

Put $y = x$. Then $y' = v + xv'$ and $y'' = 2v' + xv''$.

So (*) becomes

$$(x^2-2x+2)[2v'+xv''] - x^2[v+xv'] + x.[xv] = 0$$

$$\therefore (x^2-2x+2)[xv''+2v'] - x^3v' + [x^2v - x^2v] = 0$$

$$\therefore (x^2-2x+2)xv'' + v'[2(x^2-2x+2) - x^3] = 0$$

$$\therefore v'' + v'[\tfrac{2}{x} - x^2/(x^2-2x+2)] = 0$$

$$\therefore v'' + v'[\tfrac{2}{x} - 1 - \tfrac{(2x-2)}{x^2-2x+2}\}] = 0$$

Put $v' = w$. Then $w' + w\left[\frac{2}{x} - 1 - \frac{2x-2}{x^2-2x+2}\right] = 0$

$$\frac{dw}{dx} = \left(1 + \frac{2x-2}{x^2-2x+2} - \frac{2}{x^2}\right)w$$

$$\therefore \frac{dw}{w} = \left(1 + \frac{2x-2}{x^2-2x+2} - \frac{2}{x}\right)dx$$

$$\therefore \ln(w) = x + \ln(x^2-2x+2) - \ln(x^2) + C$$

$$\therefore w = Ae^x.(x^2-2x+2).(1/x^2) \quad \text{where } A = e^C$$

$$\therefore \frac{dv}{dx} = Ae^x.\frac{x^2-2x+2}{x^2} = A.e^x\left(1 - \frac{2}{x} + \frac{2}{x^2}\right)$$

Guess $v = Ae^x\left(1 - \frac{2}{x}\right) + B$ <u>Check</u>: $\frac{d}{dx}\left[\left(1-\frac{2}{x}\right)e^x\right]$

$\therefore y = xv = A.e^x.(x-2) + Bx$. $= \left(1-\frac{2}{x}\right).e^x + \frac{2}{x^2}e^x$ ✓

Since $e^x.(x-2)$ & x are linearly independent, this is the general solution of (*). So

$$y = A.(x-2).e^x + Bx.$$

<u>Note</u>: There is no way, in general, to find $\int e^x\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)dx$ but we guessed that $\int e^x\left(1 - \frac{2}{x} + \frac{2}{x}\right)dx = e^x\left(1 - \frac{2}{x}\right)$ and then checked that this is indeed true.

Second solution using the Theorem on p. 126 of textbook

4.1(d)#7. $\underbrace{(x^2-2x+2)}_{a_0(x)=x^2-2x+2}y'' - \underbrace{x^2}_{a_1(x)=-x^2}y' + xy' = 0 \quad \cdots (*)$, $f(x) = x$

$$v' = w = \frac{A}{[f(x)]^2}\cdot \exp\left\{-\int \frac{a_1(x)}{a_0(x)}\,dx\right\} \quad \text{(from page 126)}$$

$$= \frac{A}{x^2}\cdot \exp\left\{\int \frac{x^2}{x^2-2x+2}\,dx\right\}$$

$$= \frac{A}{x^2}\cdot \exp\left\{\int\left(1-\frac{2x-2}{x^2-2x+2}\right)dx\right\}$$

$$= \frac{A}{x^2}\cdot \exp\left\{x - \ln(x^2-2x+2)\right\}$$

$$= Ae^x \cdot \frac{1}{x^2}(x^2-2x+2) = Ae^x\left(1-\frac{2}{x}+\frac{2}{x^2}\right)$$

$\therefore \quad v = A\int e^x\left(1-\frac{2}{x}+\frac{2}{x^2}\right)dx$

Guess that $\int e^x\left(1-\frac{2}{x}+\frac{2}{x^2}\right)dx = e^x\left(1-\frac{2}{x}\right)$

Check this: $\frac{d}{dx}\left[e^x\left(1-\frac{2}{x}\right)\right] = e^x\left(1-\frac{2}{x}\right) + e^x\left[0-\left(-\frac{2}{x^2}\right)\right]$

$= e^x\left(1-\frac{2}{x}+\frac{2}{x^2}\right)$ ✓

$\therefore \quad v = A\cdot e^x\left(1-\frac{2}{x}\right) + B$

$\therefore \quad y = xv = Ax\cdot e^x\left(1-\frac{2}{x}\right) + Bx$

$= A(x-2)\cdot e^x + Bx$

is the general solution of $(*)$.