1. Euler circuits and Euler trails.
In the early 18th century there were seven bridges across various parts of the Pregel river when it passed through the city of Königsberg, Russia.

The residents of this part of Königsberg amused themselves by asking if there is way of traversing each of the seven bridges exactly once. This question is equivalent to asking if there a trail, in the multi-graph $G$ below, which uses each edge exactly once.

**Def.** Let $G$ be a graph-like object. An Euler trail of $G$ is any trail in $G$ which includes each edge of $G$ exactly once. An Euler circuit of $G$ is any closed Euler trail of $G$.

**Ex.1.** $a, b, c, d, e, b, d, a, e$ is an open Euler trail of the envelope graph on the right.
Theorem 1 (Euler's Circuit Theorem, 1735)

Let \( G \) be a connected multi-graph. Then \( G \) has an Euler circuit \( \iff \) each vertex of \( G \) is of even degree.

Proof: \((\Rightarrow)\) Suppose \( G \) has an Euler circuit \( \Rightarrow \) let's say it is \( v_0, e_1, v_1, e_2, \ldots, v_{q-1}, e_q, v_q \) where \( q = |E(G)| \). Then, starting at \( v_0 \), we traverse the circuit and delete each edge as we go along. Now observe that the degree of each vertex is reduced by 2 each time we pass through it. (In the beginning, the degree of \( v_0 \) will be reduced by 1; and in the end, it will also be reduced by 1.) When we finish traversing the circuit, each vertex will be of degree 0. Hence, at the start, the degree of each vertex must have been even.

\((\Leftarrow)\) Suppose each vertex of \( G \) is of even degree. We will show by induction on \( q \) that \( G \) has an Euler circuit.

Basis: If \( q = 0 \), then \( G \) consists of a single vertex and the trivial walk, \( v_0, v_0 \) will be an Euler circuit of \( G \).

Ind. Step: Suppose the result is true for all multi-graphs with \( \leq q \) edges. Let \( G \) be any connected multi-graph with \( q+1 \) edges and all vertices of even degree. First we find a cycle in \( G \) by starting at any vertex and keep going until a vertex for the first time, let \( v_0, v_1, \ldots, v_n, v_0 \) be the vertex sequence of \( C \). Now consider \( G - E(C) \). If \( G - E(C) \) has no edges, then \( C \) is an Euler circuit of \( G \) & we are done.
Otherwise, \( G - E(C) = G_1 U G_2 U \ldots U G_k \) where
the \( G_i \)'s are disjoint connected components of \( G - E(C) \).

Let \( v_j \) \((j = 1, \ldots, k)\) be the first vertex in \( C \)
which is a member of \( G_j \).
Now each \( G_j \) is connected & has \( \leq 9 \) edges and all vertices of even
degree. So by the induction hypothesis, each \( G_j \)
has an Euler circuit \( Q_j \). We can get an Euler
circuit of \( G \) as follows. Start at \( v_0 \), then go to
\( v_1 \) and traverse the Euler circuit \( Q_1 \), then go to \( v_2 \)
along \( C \) & traverse the Euler circuit \( Q_2 \), \ldots, then go to
\( v_k \) along \( G \) & traverse the Euler circuit \( Q_k \), and then
return to \( v_0 \) along \( C \). So if the result is true for
all connected multi-graphs with \( \leq 9 \) edges, it will be true for
all connected multi-graphs with \( 9+1 \) edges.

Conclusion: So by the Strong Principle of Mathematical
Induction it follows that the result is true for all
connected multi-graphs.
Corollary 2: Let $G$ be a connected, multigraph. Then $G$ has an open Euler trail $\iff$ exactly 2 vertices of $G$ have odd degree.

Proof: ($\Rightarrow$) Suppose $G$ has an open Euler trail. Let's say it is $v_0, e_1, v_1, e_2, \ldots, v_{2k-1}, e_k, v_k$. Put $G' = G \cup e$. Where $e$ is a new edge between $v_{2k}$ and $v_{2k+1}$. Then $v_0, e_1, v_1, e_2, \ldots, e_k, v_{2k}, e, v_{2k+1}$ will be an Euler circuit of $G'$. So each vertex of $G'$ will be of even degree. Thus $G$ must have exactly two vertices of odd degree, namely $v_0$ & $v_k$.

($\Leftarrow$) Suppose $G$ has exactly two vertices of odd degree. Let $G' = G \cup e$ where $e$ is a new edge between the two odd vertices in $G$. Then $G'$ is a connected multigraph with all vertices of even degree. So $G'$ will have an Euler circuit $Q$. Now, if we remove the edge $e$ from $Q$, this will lead us to an open Euler trail of $G$ from $v_0$ to $v_k$.

Def. Let $G$ be a digraph-like object. A directed Euler trail of $G$ is any directed trail of $G$ which includes each directed edge of $G$ exactly once. A directed Euler circuit is a closed directed Euler trail.

Theorem 3. Let $G$ be a weakly-connected digraph-like object.

(a) Then $G$ has a directed Euler circuit $\iff$ for each vertex $v$ in $G$, $\text{indeg}(v) = \text{outdeg}(v) = 0$, each $v$ is balanced.

(b) Then $G$ has an open directed Euler trail $\iff G$ has two vertices, one with an extra outdeg & the other with an extra in-degree and the rest of the vertices are balanced.
§2. Fleury's Algorithm & the Chinese Postman Problem.

Algorithm 1 (Fleury's Algorithm, 1921)

**INPUT:** A connected multi-graph $G$ with at most two vertices of odd degree

**OUTPUT:** An Euler trail of $G$.

1. Let $i \leftarrow 0$. If $G$ has odd vertices, let $v_0$ be one of the two odd vertices; otherwise let $v_0$ be any vertex of $G$. Put $G_0 \leftarrow G$ and $Q \leftarrow \langle v_0 \rangle$.

2. If $v_i$ is a pendant vertex, let $v_{i+1}$ be the only vertex adjacent to $v_i$ and $e_{i+1}$ be the edge from $v_i$ to $v_{i+1}$, and let $G_{i+1} \leftarrow G_i - \{e_{i+1}\}$; otherwise, choose any edge $e_{i+1}$ from $v_i$ to an adjacent vertex $v_{i+1}$ such that $G_i - \{e_{i+1}\}$ is still a connected graph, and let $G_{i+1} \leftarrow G_i - \{e_{i+1}\}$.

3. Let $Q \leftarrow Q \cup \langle e_{i+1}, v_{i+1} \rangle$ and $i \leftarrow i+1$. If $E(G_i) = \emptyset$, STOP; else, go to step 2.

**Ex. 1** Find an Euler circuit of the multi-graph $G$ below, by using Fleury's algorithm.

$$G = \begin{array}{c}
\text{a} \\
\text{d} \\
\text{c} \\
\text{b} \\
\text{a} \\
\text{b} \\
\text{c} \\
\end{array}$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$v_i$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a$</td>
<td>$\langle a \rangle$</td>
</tr>
<tr>
<td>1</td>
<td>$b$</td>
<td>$\langle a, e_1, b \rangle$</td>
</tr>
<tr>
<td>2</td>
<td>$d$</td>
<td>$\langle a, e_1, b, e_2 \rangle$</td>
</tr>
</tbody>
</table>
The Chinese Postman Problem.

We wish to walk which starts and ends at the Post Office, traverses each street at least once, and is of shortest possible total length. This will ensure that the postman spends the minimum amount of time covering his route.

**Def.** Let $G$ be a connected weighted multi-graph. A minimum postman walk of $G$ is a closed walk which 
contains each edge of $G$ and is of shortest possible total length.

**Ex. 1**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$u_i$</th>
<th>$Q$</th>
<th>$G_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>c</td>
<td>$(a, e_1, b, e_2, d, e_3, c)$</td>
<td>a</td>
</tr>
<tr>
<td>4</td>
<td>b</td>
<td>$(a, e_1, b, e_2, d, e_3, c, e_4, b)$</td>
<td>a</td>
</tr>
<tr>
<td>5</td>
<td>d</td>
<td>$(a, e_1, b, e_2, d, e_3, c, e_4, b, e_5, d)$</td>
<td>a</td>
</tr>
<tr>
<td>6</td>
<td>a</td>
<td>$(a, e_1, b, e_2, d, e_3, c, e_4, b, e_5, d, e_6, a)$</td>
<td>a</td>
</tr>
</tbody>
</table>

**Ex. 2**

$(a, b, c, d, e, a, e, c, a)$ is a minimum postman walk in $G$ of total length 17.
Algorithm 2 (The Postman Algorithm)

**INPUT:** A connected weighted multigraph \( G \).

**OUTPUT:** A minimum postman walk of \( G \).

1. Find all the odd vertices of \( G \) and the distances between any pair of these vertices. (There is always an even no. of odd vertices in \( G \), \( 2k \), say.)

2. Partition the set of odd vertices into pairs

\[
\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}
\]

So that \( d(u_1, v_1) + d(u_2, v_2) + \ldots + d(u_k, v_k) \) is a minimum.

3. For each \( i \) (i = 1, \ldots, k), add new edges of the shortest path from \( u_i \) to \( v_i \) to \( G \). To get a new multi-graph \( G' \). Then each vertex of \( G' \) will be of even degree.

4. Find an Euler circuit \( Q' \) of \( G' \). If we replace each of the new edges of \( Q' \) by the corresponding edge of \( G \) we will get a minimum postman walk \( W \) of \( G \).

**Example 3**

Let \( G = \)

```
  b  3  c  1  d
  a  1  2  0
  3  4
```

Odd vertices: \( b, c, e, f \)

\{b, c\} & \{e, f\}  \quad \{b, e\} \& \{c, f\}  \quad \{b, f\} \& \{c, e\}

\[3 + 4 = 7 \quad 5 + 4 = 9 \quad 1 + 3 = 4\]

\( G' = \)

```
  b  3  c  1  d
  a  1  2  0
  3  4
```

Minimum Postman walk: \( a \xrightarrow{2} b \xrightarrow{1} f \xrightarrow{1} b \xrightarrow{3} c \xrightarrow{4} e \xrightarrow{2} d \xrightarrow{1} c \xrightarrow{1} d \xrightarrow{2} e \xrightarrow{4} f \xrightarrow{3} a \). Total length = 24.
§3. Hamilton cycles & Hamilton paths.

Def. A Hamilton cycle of a graph $G$ is a cycle of $G$ which includes each vertex of $G$. A Hamilton path of $G$ is any path of $G$ which includes each vertex of $G$.

Ex. 1. Let $G = (V, E)$. Then

(a) $\langle a, b, c, d \rangle$ is a Hamilton path of $G$.

(b) $\langle a, b, d, c, a \rangle$ is a Hamilton cycle of $G$.

Def. A graph $G$ is said to be Hamilton-connected if there is a Hamilton path between any two distinct vertices of $G$.

Ex. 2. Let $G = (V, E)$. Then $G$ is Hamilton-connected.

The graph in Ex. 1 is not Hamilton-connected because there is no Hamilton path between $b$ and $c$.

Theorem 4 (Dirac, 1936). If $G$ is a graph with $p$ vertices $p \geq 3$, and $\deg(v) \geq \frac{p}{2}$ for each $v$ in $G$, then $G$ has a Hamilton cycle. (This is a special case of Prop. 5.)

Ore's Theorem

Prop. 5. Suppose $G$ is a graph with $p$ vertices, $p \geq 3$, and for each pair of non-adjacent vertices $x$ and $y$,

$$\deg(x) + \deg(y) \geq p.$$ 

Then $G$ has a Hamilton cycle.

Proof: Let $P = \langle v_1, v_2, \ldots, v_k \rangle$ be a maximal path in $G$. 


(i.e., a path which cannot be expanded to a longer path in \(G\))

We will show that the vertices \((v_1, \ldots, v_k, v_1)\) can be rearranged, if necessary, to produce a cycle \(C_1\).

Now, if \(v_k\) is adjacent to \(v_1\), then \((v_1, v_2, \ldots, v_k, v_1)\) is the vertex sequence of our cycle \(C_1\).

So suppose \(v_k\) is not adjacent to \(v_1\). Then \(\deg(v_1) + \deg(v_k) > p \geq k\). Since \(P_i\) is a maximal path in \(G\), \(v_1, v_k\) cannot be adjacent to any vertex outside of \(\{v_1, \ldots, v_k\}\). We claim that there must be an \(i \in \{3, \ldots, k-1\}\) such that \(v_i, v_i \in G\) and \(v_{i-1}, v_k \in G\). Indeed suppose this was not true. Then every time \(v_j\) is adjacent to a vertex \(v_i\), \(v_k\) cannot be adjacent to the vertex \(v_{i-1}\). So \(\deg(v_k) \leq (k-1) - \deg(v_i)\) because \(v_k\) can only be adjacent to vertices from \(\{v_2, \ldots, v_k\}\) and for every degree \(v_j\) has, \(v_k\) is denied a degree. Thus \(\deg(v_1) + \deg(v_k) \leq k - 1 \leq p - 1\). But this contradicts the fact that \(\deg(v_1) + \deg(v_k) > p\).

So \(v_i, v_i \in G\) & \(v_{i-1}, v_k \in G\) for some \(i \in \{2, \ldots, k-1\}\). Now it is easy to see that \(v_1, v_2, \ldots, v_{i-1}, v_k, v_{k-1}, \ldots, v_{i+1}, v_i, v_1\) is a cycle \(C_1\) in \(G\).
If $E(C_1) = E(G)$, then $C_1$ is a Hamilton cycle of $G$ and we are done. Otherwise, choose any vertex $x_2 \in V(G) - V(C_1)$ and let $Q_2$ be a path containing $x_2$ & all of $V(C_1)$. Then extend $Q_2$ to a maximal path $P_2$ of $G$.

Again we can arrange to get a cycle $C_2$ from the maximal path $P_2$. If $E(C_2) = E(G)$, then $C_2$ will be a Hamilton cycle of $G$ and again we will be done. Otherwise, choose a vertex $x_3 \in V(G) - V(C_2)$ and let $Q_3$ be a path containing $x_3$ & all of $V(C_2)$, and so on. In the end we will get a cycle $C_n$ of $G$ which is a Hamilton cycle of $G$ because $G$ is a finite graph.

Ex. 3 Let $G = a \rightarrow b \rightarrow f$. Then

\[ P_1 = a \rightarrow d \rightarrow c \rightarrow e \rightarrow b, \quad C_1 = a \rightarrow b \rightarrow a, \quad x_2 = f \]

\[ Q_2 = a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow a \rightarrow \quad P_2 = a \rightarrow b \rightarrow f \rightarrow d \rightarrow c \rightarrow e \]

\[ C_2 = a \rightarrow b \rightarrow f \rightarrow c \rightarrow d \rightarrow e \]
Corollary 6: Let $G$ be a graph with $p$ vertices such that for any pair of non-adjacent vertices $x$ and $y$,
\[ \text{deg}(x) + \text{deg}(y) \geq p - 1. \]
Then $G$ has a Hamilton path.

Proof: Let $H$ be the graph obtained by adding a new vertex $v_{p+1}$ and edges from $v_{p+1}$ to each of the vertices of $G$.

Then $H$ has $p+1$ vertices and for any pair of non-adjacent vertices $x$ and $y$ in $H$, we have
\[ \text{deg}_H(x) + \text{deg}_H(y) = \left\{ \text{deg}_G(x) + 1 \right\} + \left\{ \text{deg}_G(y) + 1 \right\} \]
\[ \geq (p - 1) + 2 = p + 1. \]

So by Prop 5, $H$ has a Hamilton cycle $C$. Now if we delete the vertex $v_{p+1}$ from $C$, we will get a Hamilton path $P$ of $G$.

Prop 7: Let $G$ be a graph with $p$ vertices such that for any pair of non-adjacent vertices $x$ and $y$, \[ \text{deg}(x) + \text{deg}(y) \geq p + 1. \]
Then $G$ is Hamilton-connected.

Proof: Do for homework.
The Travelling Salesman problem.

We want a closed walk of shortest possible length which includes each city at least once.

Def. Let G be a weighted graph. A minimum salesman walk is a closed walk which includes each vertex of G and is of the shortest possible length.

Ex. 4 Let \( G = a \xrightarrow{4} b \xrightarrow{1} \ldots \xrightarrow{2} \ldots \xrightarrow{2} a \). Then

\[
a^2 e^6 b^1 c^1 d^2 e^2 a
\]

Total length = 9

EDGE TRAVERSAL PROBLEMS VERTEX TRAVERSAL PROBLEMS
1. Euler circuit Hamilton cycle
2. Open Euler trail Hamilton path
3. Euler’s Circuit theorem No corresponding theorem
4. Euler’s Algorithm No corresponding algorithm
5. Chinese Postman problem Travelling Salesman problem
6. Minimum Postman walk Minimum Salesman walk
7. Postman Algorithm No corresponding algorithm