Ch. 6 - Planar graphs

8.1. Euler's Planarity formula & other properties of planar graphs

Recall that a graph $G = (V, E)$ can be represented geometrically as a subset of the plane $\mathbb{R}^2$ by using small disks for the vertices in $V$ and arcs joining two disks to represent the edges of $E$.

Def. A graph $G$ is said to be planar if we can find a representation of it in the plane in which no two edges intersect. Such a representation $E(G)$ is called a planar embedding of $G$.

Ex. 1. Let $G = K_4 = a \quad b$. Below are two planar embeddings of $G$.

\[ \begin{align*}
E_1(G) & \quad \quad \quad \quad \quad \quad E_2(G) \\
\end{align*} \]

Def. Let $E(G)$ be a planar embedding of $G$. Then $\mathbb{R}^2 - E(G)$ will be a union of a finite number of connected open subsets of $\mathbb{R}^2$. Each of these open connected subsets is called a region of $E(G)$.

Ex. 2. The four regions of $E_1(G)$ from Ex. 1.
Although the size and orientation of the regions depend on the embedding \( E(G) \), the number of regions depends only on \( G \) and not on the particular embedding, as we will shortly see.

**Theorem 1 (Euler’s planarity formula)**

Let \( r_e(G) \) = the number of regions into which the planar embedding \( E(G) \) partitions \( R^2 \). If \( G \) is connected, then \( r_e(G) = |E(G)| + 2 - |V(G)| \), for any \( \varepsilon \).

**Proof:** We will prove the result by parametric induction on \( q = |E(G)| \). First, fix \( p = V(G) \).

Since \( G \) is connected, \( q \geq p - 1 \).

**Basis:** If \( q = p - 1 \), then \( G \) is a connected graph with \( p - 1 \) edges. So \( G \) must be a tree and hence \( r_e(G) = 1 \), for any planar embedding \( E(G) \). Since \( 1 = (p - 1) + 2 - p \) it follows that \( r_e(G) = |E(G)| + 2 - |V(G)| \). So the result is true for \( q = p - 1 \) and for any \( \varepsilon \).

**Ind. Step:** Suppose the result is true for all graphs with \( q \) edges (where \( q \geq 1 \)) and for any \( \varepsilon \). Let \( G \) be a connected graph with \( q + 1 \) edges and \( E(G) \) be any planar embedding of \( G \). Since \( G \) has \( q + 1 \) edges & \( q \geq p - 1 \), \( G \) cannot be a tree. So \( G \) must have at least one cycle, \( C \) say. Let \( e \) be any edge in \( C \) and put \( G' = G - \{e\} \). Since the removal of \( e \) will reduce the number of regions
of $E(G)$, we have $r_e(G') = r_e(G) - 1$ and $|E(G')| = |E(G)| - 1$. Also $V(G) = V(G')$ and $r_e(G') = |E(G')| + 2 - |V(G')|$ by the induction hypothesis. So

$$r_e(G) = r_e(G') + 1$$

$$= |E(G)| + 2 - |V(G)| + 1$$

$$= |E(G)| + 2 - |V(G)|.$$ 

So, if the result is true for $g$, it will be true for $g+1$. Hence, by the Principle of Mathematical Induction, the result is true for all $g$. Since $p$ was arbitrary, it is also true for all $p$. Hence the result is true for all planar graphs.

Notation: Since $r_e(G)$ does not depend on the particular embedding $e(G)$ that we use, we will denote it by just $r(G)$. We will also use $q(G)$ for $|E(G)|$ and $p(G)$ for $|V(G)|$.

**Corollary 2 (Euler's Generalized Planarity Formula)**

Let $G$ be any planar graph & $k = \text{number of connected components of } G$. Then:

$$r(G) = q(G) + (k+1) - p(G).$$

**Proof:** Let $G_1, \ldots, G_k$ be the $k$ connected components of $G$ and $E(G)$ be any planar embedding of $G$. Then for each $i = 1, \ldots, k$:

$$r(G_i) = q(G_i) + 2 - p(G_i).$$
So \[ \sum_{i=1}^{k} r(G_i) = \sum_{i=1}^{k} q(G_i) + \sum_{i=1}^{k} e - \sum_{i=1}^{k} p(G_i). \tag{4} \]

But the infinite region is counted \( k \) times (instead of just once) in the sum \( \sum_{i=1}^{k} r(G_i) \). So
\[ r(G) + (k-1) = q(G) + 2k - p(G). \]
Hence \( r(G) = q(G) + (k+1) - p(G). \)

**Def.** A maximal planar graph is any planar graph \( G \) such that \( G \cup \{x,y\} \) is non-planar for any pair of non-adjacent vertices \( x \) & \( y \) in \( G \).

**Ex.3** (a) \( K_3 \) & \( K_4 \) are maximal planar graphs.
(b) \( K_5 - \{ \text{any edge}\} \) is a maximal planar graph
(c) \( K_{2,3} \) is not a maximal planar graph.

**Prop.3** Let \( G \) be a maximal planar graph with \( p \geq 3 \) vertices and \( E(G) \) be any planar embedding of \( G \). Then each region of \( E(G) \) is bounded by 3 edges.

**Proof:** Suppose \( E(G) \) has a region which is bounded by \( >4 \) edges. Then we can find a region
\[ v_1, v_2, v_3, \ldots, v_n, v_1, \]
which is bounded by a cycle \( C = \langle v_1, v_2, v_3, \ldots, v_n, v_1 \rangle \).
There are two cases:

Case (i) \( v_1, v_3 \in E(G) \). In this case, the embedding of the edge \( v_1, v_3 \) must be outside the cycle \( C \). But this means that \( v_1, v_3 \) is prevented from being an edge in \( E(G) \). So we can embed the edge \( v_1, v_3 \) inside the cycle \( C \) and hence contradict the fact that \( G \) is maximal planar.

Case (ii) \( v_1, v_3 \notin E(G) \). In this case \( v_1, v_3 \) are non-adjacent vertices and we can embed \( v_1, v_3 \) inside the cycle \( C \) — thereby contradicting the fact that \( G \) is maximal planar again. Hence every region of \( E(G) \) is bounded by 3 edges.

Prop 4. Let \( G \) be a graph with \( p \geq 3 \) & \( q = E(G) \).

(a) If \( G \) is maximal planar, then \( q = 3p - 6 \).

(b) If \( G \) is planar, then \( q \leq 3p - 6 \).

Proof (a) Suppose \( G \) is maximal planar. Let \( E(G) \) be any planar embedding of \( G \) & \( r = r(G) \). Let \( A_1, \ldots, A_r \) be the regions of \( E(G) \). Since each region of \( A_i \) is bounded by 3 edges, \( 3r = e(A_1) + \ldots + e(A_r) = \text{number of edges counted} = 2q \) because each edge was counted exactly 2 times.

So \( 3r = 2q \). But \( r = q + 2 - p \) by Euler's planarity formula. Hence \( 3(q + 2 - p) = 2q \).

\[ 3q + 6 - 3p = 2q \quad \Rightarrow \quad q = 3p - 6 \]

(b) Let \( G \) be a planar graph. If we add edges one at a time to \( G \), we will get a maximal planar graph \( G' \). So \( q = q(G) \leq q(G') = 3p(G') - 6 = 3p(G) - 6 \). \( \therefore q \leq 3p - 6 \).
§2 Non-planar graphs & Kuratowski's theorem.

Corollary 5: $K_5$ is a non-planar graph.

Proof: Suppose $K_5$ was planar. Then by Prop 4(1) we will get $\Delta(K_5) \leq 3p(K_5) - 6$. Since $K_5$ has 10 edges and 5 vertices, this means that $10 \leq 3(5) - 6$. So $10 \leq 9$ which is a contradiction. Hence $K_5$ is non-planar.

Prop 6: If $G$ is a planar bipartite graph, then $\Delta(G) \leq 2p(G) - 4$.

Proof: Let $E(G)$ be a planar embedding of $G$ and $r = \Delta(G)$. Since $G$ is a bipartite graph, each cycle of $G$ must have an even number of edges. Since $G$ is a graph, we need at least 3 edges to form a cycle. So each region of $E(G)$ will be bounded by a cycle with at least 3 edges. Let $A_1, \ldots, A_r$ be the regions of $E(G)$. Then $e(A_i) \geq 4$ for each $i$. So $4r = 4 + 4 + \cdots + 4 \ (r \text{ times})$ $\leq e(A_1) + e(A_2) + \cdots + e(A_r)$ $= \text{number of edges counted} = 2\Delta$. So $2r \leq \Delta$. But $\Delta = 2p - r$. Hence $2(2p - r) \leq \Delta \Rightarrow 2(2p - r) \leq 2r$ $\Rightarrow 2p - 4 \leq q$ $\Rightarrow q \leq 2p - 4$.

Corollary 7: $K_{3,3}$ is a non-planar graph.

Proof: Suppose $K_{3,3}$ was planar. Then by Prop 6, $\Delta(K_{3,3}) \leq 2p(K_{3,3}) - 4$. So $9 \leq 2(6) - 4$, i.e., $q \leq 8$ which is a contradiction. Hence $K_{3,3}$ is non-planar.
Q1. When exactly is a graph non-planar?

Ans. We know that if \( q(G) > 3p(G) - 6 \), then \( G \) is non-planar. Also, if \( G \) is bipartite & \( q(G) > 2p(G) - 4 \), then \( G \) is also non-planar. But if \( q(G) \leq 3p(G) - 6 \), it does not follow that \( G \) is planar. Also, if \( G \) is bipartite & \( q(G) \leq 2p(G) - 4 \), it does not follow that \( G \) is planar.

Ex. 1

(a) \( G_1 = \)

(b) \( G_2 = \)

\[ q(G_1) = 11 \leq 3(6) - 6 = 3p(G_1) - 6 \]
\[ q(G_2) = 10 \leq 2(7) - 4 = 2p(G_2) - 4 \]

It is easy to see that if \( G_1 \) & \( G_2 \) were planar, then \( K_5 \) & \( K_{3,3} \) will be also be planar. So \( G_1 \) & \( G_2 \) are non-planar.

Since \( K_5 \) & \( K_{3,3} \) are non-planar, any graph \( G \) that contains \( K_5 \) or \( K_{3,3} \) as a subgraph (or something that "amounts" to being a subgraph) will be non-planar. So \( K_6, K_7, K_8, \ldots \) and \( K_{3,4}, K_{3,5}, \ldots, K_{4,4}, K_{4,5}, \ldots, K_{5,5} \) are all non-planar.

Q2. (a) Is \( K_{2,3} \) planar? Yes
(b) Is \( K_{2,2,2} \) planar? Yes, do for H.W.
(c) Is \( K_{2,2,3} \) planar? No, \( 16 \neq 3(7) - 6 \)
(d) Is \( K_{2,2,2,2} \) planar? No, \( 24 \neq 3(8) - 6 \)
Def. Let $e = uv$ be an edge in a graph $G$. Then we can create a vertex of degree 2 on the edge $e$ by adding a new vertex $x$ to $G$, by adding the edges $ux$ & $xv$, and by deleting the edge $uv$ from $G$.

Ex. 2

\[ G = \begin{array}{c}
\text{u} \\
\text{w} \\
\text{y} \\
\text{o}_3 \\
\text{v}
\end{array} \quad \rightarrow \quad G' = \begin{array}{c}
\text{u} \\
\text{w} \\
\text{y} \\
\text{o}_3 \\
\text{x} \\
\text{v}
\end{array} \]

Def. Let $x$ be a vertex of degree 2 in a graph $G$. Then we can merge out the vertex $x$ from $G$ by deleting the vertex $x$ and by adding a new edge between the two vertices that were adjacent to $x$ in $G$.

Ex. 3

\[ G = \begin{array}{c}
\text{u} \\
\text{w} \\
\text{y} \\
\text{o}_3 \\
\text{v}
\end{array} \quad \rightarrow \quad G' = \begin{array}{c}
\text{u} \\
\text{w} \\
\text{y} \\
\text{o}_3 \\
\text{v}
\end{array} \]

Def. Two graphs $G$ & $H$ are homeomorphic if we can transform $G$ into $H$ by creating vertices of degree 2 on certain edges of $G$ or by merging out certain vertices of degree 2 in $G$.

Theorem 8 (Kuratowski's planarity theorem). $G$ is planar $\iff G$ has no subgraph which is homeomorphic to $K_5$ or $K_{3,3}$.

Proof: ($\Rightarrow$) Suppose $G$ is planar. Then $G$ cannot contain any subgraph which is homeomorphic to $K_5$ or $K_{3,3}$ (otherwise $K_5$ or $K_{3,3}$ would be planar). ($\Leftarrow$): hard - see textbook.
§3. The Demoucron, Malgrange & Perruisel Planarity Algorithm

Def. Let $G$ be a graph and $H$ be a subgraph of $G$. A piece of $G$ relative to $H$ is either
(a) an edge $e = uv$ with $e \notin E(G)$ & $u, v \in V(H)$ or
(b) a component $C$ of $G - V(H)$ plus all the edges joining vertices of $C$ to vertices of $H$.

Def. Let $P$ be a piece of $G$ relative to $H$. If $v \in V(P) \cap V(H)$, we say that $v$ is a contact vertex of $P$. If the piece $P$ has 2 or more contact vertices, we say that $P$ is a segment of $G$ relative to $H$.

Ex.1. Let $G = \begin{array}{ccc} a & b & c \\ o & - & d \\ e & f & g \end{array}$ and $H = \begin{array}{ccc} e & c & d \\ g & - & f \\ e \end{array}$

Then $G - V(H) = \begin{array}{ccc} a & b & o \\ g \end{array}$. So the pieces of $G$ relative are as shown below.

- segment
- not a segment

Def. Recall that a cut-vertex of $G$ is any vertex $v$ of a connected graph such that $G - \{v\}$ is disconnected. A connected graph with no cut-vertex is called a block (or connected block).

Ex.2. $G = \begin{array}{ccc} a & b & c \\ \circ & \circ & \circ \\ b & c & b \end{array}$ blocks of $G$: $\begin{array}{ccc} a & c & b \\ \circ & \circ & \circ \end{array}$
The DMP Planarity algorithm will take a connected block as input. So before we apply the algorithm we must first pre-process the graph we are testing for planarity.

Pre-processing $G$ for the DMP Planarity Algorithm

1. If $G$ is not connected, then consider each component separately.
2. If a connected component has cut-vertices, then split the cut-vertices to get a set of blocks of $G$.
3. If $q(G_i) > 3p(G_i) - 6$ for any block $G_i$ with $p(G_i) \geq 3$, then that block is non-planar & so $G$ is non-planar.

Ex. 3 Let $G = [...]

Then the 4 blocks of $G$ are shown below:

Algorithm 1 (The DMP Planarity algorithm)

INPUT: A pre-processed block $G = (V, E)$

OUTPUT: 

- A planar embedding of $G$, if $G$ is planar
- NON-PLANAR, if $G$ is non-planar.

1. If $G$ has no cycles, then $G$ must be the tree $K_2$, and $\cdots$ is a planar embedding of $G$ & we are done.

Otherwise, choose any cycle $C$ in $G$, let $i=1$, $r\leq 2$, and $H_i \leftarrow$ a planar embedding of $C$.
2. If \( E(H_i) = E(G) \), STOP. Otherwise, find all the segments of \( G \) relative to \( H_i \) and for each segment \( S \), let \( R_i(S) \subseteq \) the set of regions of \( H_i \) into which \( S \) can be compatibly embedded.

3. If there is a segment \( S \) such that \( R_i(S) = \emptyset \), then say NON-PLANAR and STOP;
If there is a segment \( S \) such that \( |R_i(S)| = 1 \), then let \( R \subseteq \) the unique region in \( R_i(S) \);
Otherwise, choose any segment \( S \) and let \( R \subseteq \) any one of the regions in \( R_i(S) \).

4. Choose any path \( L \) in \( S \) which connects two contact vertices of \( S \). Then let \( H_{i+1} = H_i \cup \{L\} \) the embedding of \( L \) in the region \( R_i \).
\( i \leftarrow i+1, \ \tau \leftarrow \tau +1 \), and go to step 2.

**Ex. 3** Determine whether or not the graph on the right is planar.

**Sol** 
\[ H_1 = \]

\[ H_2 = \]

**Segments of \( G \) relative to \( H_i \):**

\[ R_i(S): \{1,2\} \]
\[ \{1,2\} \]
\[ \{1,2,3\} \]
\[ \{3\} \]
$$H_3 = \begin{array}{c}
  a & b & c \\
  f & e & d
\end{array}$$

$$R_3(5) = \{3\} \uparrow R$$

$$H_4 = \begin{array}{c}
  a & b & c \\
  f & g & d
\end{array}$$

$$R_4(5) = \emptyset \quad \text{STOP. NON-PLEANR}$$

Ex. 4: Determine whether or not $K_{2,2,2}$ is planar.

Sol. $H_1 = \begin{array}{c}
  a & b & c \\
  f & e & d
\end{array}$

Segments of G rel. to $H_1$

$K_{2,2,2}$ is planar
§ 4. Polyhedral graphs & the geometric dual

A polyhedron is a solid figure with plane polygonal faces that can be continuously distorted (transformed) into a solid sphere.

Ex. 1 Some polyhedra (some textbooks call these simple polyhedra)
- Tetrahedron
- Cube
- Square based pyramid

Ex. 2 Some solids that are not polyhedra
- Two tetrahedra joined at a vertex.
- Large cube with a smaller cube hollowed out in the center.
- The picture frame (solid cube with a hole drilled through front to back face)
- Two tetrahedra welded together along an edge.
A polyhedral graph is any graph that can be obtained by considering the vertices and edges of a polyhedron as the vertices and edges of a graph.

**Ex. 3**

![](tetrahedral_graph.png)

**Prop 3.** If $G$ is a polyhedral graph, then $G$ is planar and obviously connected.

If $G$ is a polyhedral graph, consider the polyhedron from which $G$ was obtained. If we imagine that the polyhedron is hollow and we make a hole in one face and stretch the polyhedron onto the plane, we will get a planar embedding of $G$.

![](embedding.png)

**Def.** A regular polyhedron is one in which each face is a fixed regular polygon and in which each vertex has the same no. of edges incident to it.

**Ex. 4.** The tetrahedron and cube are regular.
Qu: Is a regular polyhedron? (assume each face is an equilateral triangle) — (No)
Is it even a polyhedron? (Yes).

Theorem 9: There are exactly 5 regular polyhedra.

Proof: Suppose P is a regular polyhedron. Then each face of P is a regular polygon with k edges, say. Let \( A_1, \ldots, A_r \) be the faces of P. Then
\[
e(\overline{A_1}) + \ldots + e(\overline{A_r}) = 2q
\]
So
\[
k \cdot r = 2q \quad \ldots \quad (1)
\]
Also, each vertex has a fixed number, \( l \) say, of edges incident to it. So, the degree of each vertex is \( l \). Now by the first theorem of graph Th., sum of degrees = \( 2 \) (no. of edges)
So
\[
l \cdot p = 2q \quad \ldots \quad (2)
\]
Also by Euler’s formula \( r = q + 2 - p \quad \ldots \quad (3) \)

Now from (1) we have \( r = \frac{2q}{k} \), and from (2) we have \( p = \frac{2q}{l} \). Substituting in (3), we get
\[
\frac{2q}{k} = q + 2 - \frac{2q}{l}
\]
So
\[
\frac{2q}{k} + \frac{2q}{l} = \frac{2q}{2} + \frac{2q}{2}
\]
Hence
\[
\frac{1}{k} + \frac{1}{l} = \frac{1}{2} + \frac{1}{2} \quad \ldots \quad (4)
\]
Now we know that \( k \geq 3 \) (a polygon can't have less than 3 edges) and \( l \geq 3 \) (for the figure to be solid we need at least 3 edges at each vertex), and from (4) we get \( \frac{k + \frac{1}{2}}{l} > \frac{1}{2} \). So the only possible values of \( k \) and \( l \) are given in the table below.

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\( \)
§5. The geometric dual & graphs on other surfaces

Def. Let \( \mathcal{E}(G) \) be a planar embedding of a planar graph \( G \). We define the geometric dual \( G^* \) of \( G \) by

(a) \( V(G^*) \) = set of the regions into which \( \mathcal{E}(G) \) partitions the plane \( \mathbb{R}^2 \).

(b) For each edge \( e \) in \( \mathcal{E}(G) \) that is a common boundary of the regions \( R_1 \) & \( R_2 \), we get an edge between \( R_1 \) & \( R_2 \) in \( G^* \).

Ex. 1. Let \( \mathcal{E}(G) = \)

\[
\begin{array}{c}
R_3 \\
\downarrow \\
R_2 \\
\downarrow \\
R_1 \\
\end{array}
\]

Then \( G^* = \)

\[
\begin{array}{c}
R_3 \\
\downarrow \\
R_2 \\
\downarrow \\
R_1 \\
\end{array}
\]

Ex. 2. Let \( \mathcal{E}(G) = \)

\[
\begin{array}{c}
R_3 \\
\downarrow \\
R_1 \\
\end{array}
\]

Then \( G^* = \)

\[
\begin{array}{c}
R_3 \\
\downarrow \\
R_1 \\
\end{array}
\]

Note

From Ex. 1, it is clearly that \( G^* \) depends on the particular embedding of \( G \) that is selected. In general, \( G^* \) will be a multi-pseudo-graph.

Q.1. When is \( G^* \) independent of the embedding \( \mathcal{E} \) ?

Q.2. When is \( G^* \) guaranteed to be a graph?

Def. The graph \( G \) is \( 1 \)-isomorphic to \( H \) if we can split \( G \) into blocks (by splitting its cut-vertices) and refit the blocks (by identifying pairs of vertices) to get \( H \).
Ex. 2 Let $G = \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
\circ \circ \circ
\end{array}
\end{array}$ Blocks of $G$:

& $H = \begin{array}{c}
\begin{array}{c}
\circ \circ \\
\circ \circ
\end{array}
\end{array}$ Blocks of $H$:

Then $G$ is 1-isomorphic to $H$.

Def. The graph $G$ is 2-isomorphic to $H$ if by flipping around a portion of $G$ which can be separated by splitting two vertices we can get a graph $G'$ which is 1-isomorphic to $H$.

Ex. 3 Let $G = \begin{array}{c}
\begin{array}{c}
\circ \\
\circ
\end{array}
\end{array}$

Then $G' = \begin{array}{c}
\begin{array}{c}
\circ \\
\circ
\end{array}
\end{array}$

& if $H = \begin{array}{c}
\begin{array}{c}
\circ \\
\circ
\end{array}
\end{array}$, then $G$ is 2-isomorphic to $H$.

Theorem 10: If $G$ is a planar graph with $k_r(G) \geq 3$, then

(a) $G^e$ will be independent of the embedding $\mathcal{E}$

(b) $G^e$ will always be a graph

(c) $(G^e)^* \cong G$. 
Def. Let $G$ be a graph with $k_v(G) \geq 3$. Then $G^*$ does not depend on $E$. So we will denote $G^*$ by $G^*$. We say that $G^*$ is self-dual if $G^* \cong G$.

Ex. 4. Let $G = K_4$. Then $G^* = \text{Diagram}$.
So $K_4$ is self-dual.

Ex. 5. Let $G = \text{Diagram}$. Then $G^* = \text{Diagram}$.
And $(G^*)^* \cong G$. So the geometrical dual of the cube graph is the octahedral graph & the geometrical dual of the octahedral graph is the cube graph.

Ex. 6. Show that $(\text{icosahedral graph})^* = \text{dodecahedral graph}$ and $(\text{dodecahedral graph})^* = \text{icosahedral graph}$.

Graphs on other surfaces

1. A graph that can be embedded (with no edges crossing) on the surface of a sphere is called a spheroidal graph.
Fact: $G$ is spheroidal $\iff G$ is spheroidal.

2. A graph that can be embedded (with no edges crossing) on the surface of a torus is called a toroidal graph.
Fact: $K_5$, $K_{3,3}$, $K_{3,4}$, $K_6$ & $K_7$ are all toroidal graphs.