Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.

(15) 1. Find the distances from b to each of the other vertices of the graph on the right by using Dijkstra's Algorithm.

(20) 2(a) Find a graph with degree sequence 5,3,3,3,3,3 by using the graphical sequence algorithm.

(b) Find a minimal spanning tree of the graph on the right by using Prim's Algorithm & starting at b.

(20) 3. (a) Find the tree corresponding to \((2,1,2,6)\) via Prüfer's Tree Decoding Algorithm.

(b) The six characters a,b,c,d,e,f occur with frequencies 25,4,6,15,40,10 respectively. Find an optimal binary coding for these six characters.

(15) 4. (a) Define what is a connected component of a disconnected graph G.

(b) Prove that if G is a disconnected graph, then \(G^c\) is forced to be connected.

(15) 5. (a) Define what is an pendant vertex of a graph G.

(b) Prove that in any tree T the number of leaves is equal to 
\[2 + \sum_{\deg(v) > 2} (\deg(v) - 2)\]

(15) 6. (a) Define what is a legal flow \(f\) in a network \(N\) and what is the value of the flow \(f\).

(b) Prove that there are only two trees \(T\) such that \(T^c\) is also a tree.
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### MAD 3305 - Graph Theory

#### Solutions to Test #1

**Florida Internat'l Univ.**  
**Spring 2007.**

<table>
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<th>L(b)</th>
<th>L(c)</th>
<th>L(d)</th>
<th>L(e)</th>
<th>L(f)</th>
<th>L(g)</th>
<th>T</th>
<th>( \nu )</th>
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<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
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<tr>
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<td>3</td>
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<td>( \infty )</td>
<td>25</td>
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<td>4</td>
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<td>e</td>
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<td>6</td>
<td>10</td>
<td>12</td>
<td>12</td>
<td>( \infty )</td>
<td>( {a, c, f, g} )</td>
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<tr>
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<td>7</td>
<td>9</td>
<td>( {c, f, g} )</td>
<td>( {c} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>10</td>
<td>9</td>
<td>( {c} )</td>
<td>( {} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ d(b, \cdot) = 7 \]

\[ d(U, \cdot) = \begin{array}{c|cccccccc}
\emptyset & \{b\} & \{b, c\} & \{b, a, f\} & \{b, a, f, c\} & \{b, a, f, c, d\} & \{b, a, f, c, d, e\} & \{b, a, f, c, d, e, \} \\
\hline
\{b\} & \{b\} & \{b, a\} & \{b, a, f\} & \{b, a, f, c\} & \{b, a, f, c, d\} & \{b, a, f, c, d, e\} & \{b, a, f, c, d, e, \} \\
\{b, a\} & \{b, a\} & \{b, a, f\} & \{b, a, f, c\} & \{b, a, f, c, d\} & \{b, a, f, c, d, e\} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} \\
\{b, a, f\} & \{b, a, f\} & \{b, a, f, c\} & \{b, a, f, c, d\} & \{b, a, f, c, d, e\} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} \\
\{b, a, f, c\} & \{b, a, f, c\} & \{b, a, f, c, d\} & \{b, a, f, c, d, e\} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} \\
\{b, a, f, c, d\} & \{b, a, f, c, d\} & \{b, a, f, c, d, e\} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} \\
\{b, a, f, c, d, e\} & \{b, a, f, c, d, e\} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} & \{b, a, f, c, d, e, \} \\
\end{array} \]

\[ \omega(T_{\text{min}}) = 21 \]

\[ T_{\text{min}} = \begin{array}{cccccccc}
\emptyset & a & f & e & a & s & d & \emptyset \\
\hline
\{b\} & \{b\} & \{b, c\} & \{b, a\} & \{b, a, f\} & \{b, a, f, c\} & \{b, a, f, c, d\} & \{b, a, f, c, d, e\} \\
\end{array} \]
(a) A connected component of $G$ is a maximal connected subgraph $H$ of $G$. $H$ is maximal in the sense that there is no connected subgraph $H'$ which strictly contains $H$.

(b) Suppose $G$ is disconnected. Let $u$ & $v$ be any two vertices in $G^c$. Now if $uv \notin G$, then $uv \in G^c$ and we instantly get a path $u-v$ from $u$ to $v$ in $G^c$. And if $uv \in G$, then there must exist a component $H_2$ of $G$ that does not contain $u$ & $v$. Choose a vertex $w$ in $H_2$. Then $uw$ & $wv$ will both be edges in $G^c$. So $u-w-v$ will be a path from $u$ to $v$ in $G^c$. So in either case we get a path from $u$ to $v$ in $G^c$. Since $u$ and $v$ are arbitrary, it follows that $G^c$ is a connected graph.
5(a) A pendant vertex of \( G \) is any vertex of degree 1.

(b) Let \( p = |V(T)| \), \( k = \max \) degree in \( T \), and \( n_i \) = number of vertices in \( T \) with degree \( i \). Then
\[
n_1 + n_2 + n_3 + \cdots + n_k = p.
\]
Also sum of degrees in \( T \)
\[
= 2(p-1).
\]
So:
\[
1. n_1 + 2n_2 + 3n_3 + \cdots + k \cdot n_k = 2p - 2
\]
\[
2. n_1 + 2n_2 + 3n_3 + \cdots + k \cdot n_k = (2n_1 + 2n_2 + 3n_3 + \cdots + 2n_k) - 2
\]
\[
3. 2 + 0n_2 + (3-2)n_3 + (4-2)n_4 + \cdots + (k-2)n_k = 2n_1 - n_1
\]
\[
= 2 + \sum_{\text{deg}(v) > 2} \{\text{deg}(v) - 2\}
\]

6(a) A legal flow in a network \( N \) is any function \( f: E \rightarrow [0, \infty) \)
such that (i) \( f(e) \leq c(e) \) for each \( e \in E \), and
(ii) \( \sum_{e \in \text{Out}(v)} f(e) = \sum_{e \in \text{In}(v)} f(e) \) for each \( v \in V \setminus \{s,t\} \).
The value of the flow is defined by \( F(f) = \sum_{e \in \text{In}(t)} f(e) \).

(b) Suppose \( T \) is a tree such that \( T^c \) is also a tree.

Let \( p = |V(T)| \). Then \( |E(T)| = |E(T^c)| = p - 1 \).
Also \( |E(T)| + |E(T^c)| = |E(K_p)| = p(p-1)/2 \).

So:
\[
(p-1) + (p-1) = p(p-1)/2
\]
\[
(p-1) = p(p-1)
\]
\[
(4 - p)(p-1) = 0 \Rightarrow p = 1 \text{ or } 4.
\]
So there can only be two possible values of \( p \).

If \( p = 1 \), then \( T = \square \) and \( T^c = \bigcirc \).
If \( p = 4 \), then \( T = \bigcirc \) and \( T^c = \bigcirc \).

So there are only two trees \( T \) such that \( T^c \)
is also a tree.

Note: When \( p = 4 \), \( T \) cannot be \( \bigcirc \) bec. \( T^c = \bigcirc \) is not a tree.