TEST #2 - SPRING 2014

Answer all 6 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(15) 1. Find a maximal flow $f'$ in the network on the right by using the Ford-Fulkerson algorithm. Also find the source-separating set of vertices $S'$ corresponding to $f'$.

(15) 2. Find a minimum postman walk of the graph on the right by using the Postman algorithm; and the total length of your minimum postman walk?

(18) 3. Determine whether or not the graph on the right is planar by using the DMP Planarity algorithm. [Show the embeddings for each step of the algorithm.]

(22) 4 (a) Find $P_G(\lambda)$ for the graph $G$ on the right by using the Chromatic Polynomial algorithm.
(b) Prove that $P_T(\lambda) = \lambda(\lambda-1)^{n-1}$ for any tree $T$ with $n$ vertices.

(15) 5 (a) Define what is a minimum salesman walk in a graph $G$.
(b) Use Ore's Theorem to prove that if $G$ is a graph with $p$ vertices and $\text{deg}(x) + \text{deg}(y) > p - 1$ for any pair of non-adjacent vertices $x$ & $y$ in $G$, then $G$ has a Hamilton path.

(15) 6 (a) Define what is a simple polyhedron.
(b) Let $G$ be a simple polyhedron with no triangular faces. Prove that $q \leq 2p - 4$.
[You may use any theorem proved in class for Qu. #6]
1(a) 1st augmenting semi-path:
\[ S \xrightarrow{9} a \xrightarrow{0.9} b \xrightarrow{0.9} t \]
Slacks: 9 7 9 \( M_1 = 7 \)

2nd augmenting semi-path:
\[ S \xrightarrow{(0,3)} b \xrightarrow{7.9} t \]
Slacks: 3 2 \( M_2 = 2 \)

3rd augmenting semi-path:
\[ S \xrightarrow{(0,6)} c \xrightarrow{(0,6)} d \xrightarrow{(0,8)} t \]
Slacks: 6 6 8 \( M_3 = 6 \)

\( \forall \mathbf{f}^* \) = net flow into \( t \) = \( f^*(bc) + f^*(dt) = 9 + 6 = 15 \)
\( S^* = \{ u \in V(G) : \text{there is an aug. semi-path from} s \text{ to } u \} = \{ s, a, b, c \} \)
\( c(S^*) = \text{sum of outward capacities} = c(cd) + c(bt) = 6 + 9 = 15 \).

2(a) Odd vertices:\{a, b, d, f\}

Minimum Postman walk = \[ a \xrightarrow{4} b \xrightarrow{2} c \xrightarrow{4} d \xrightarrow{1} f \xrightarrow{2} e \xrightarrow{1} c \xrightarrow{3} b \xrightarrow{1} e \xrightarrow{1} a \].

Total length = 21 = 18 + 3.

3. Embedding of Hi

Segments of G relative to Hi:

\[ \{1,2\} \]
4 (a) 

\[ P_G(\lambda) = P_{K_e}(\lambda) + 2P_{K_3}(\lambda) \]

\[ = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) \]

\[ = \lambda(\lambda-1)(\lambda-2)[(\lambda-3)+2] = \lambda(\lambda-1)^2(\lambda-2). \]

(b) We will prove the result by induction on \( n = |V(T)| \).

If \( n=1 \), then \( T \cong K_1 \), so \( P_T(\lambda) = \lambda = \lambda(\lambda-1)^0 \). Hence the result is true for \( n=1 \). Now suppose the result is true for all trees with \( n \) vertices. Let \( T \) be any tree with \( n+1 \) vertices and choose any leaf \( v_0 \) in \( T \). (To get a leaf just look at the endpoints of a longest path in \( T \).)

Put \( T' = T - \{v_0\} \). Then \( P_{T'}(\lambda) = (\lambda-1).P_T(\lambda) \)

\[ = (\lambda-1).\lambda(\lambda-1)^{n-1} = \lambda(\lambda-1)^{n+1-1} \]

by the induction hypothesis. So if the result is true for \( n \), it will be true for \( n+1 \). Hence the result is true for all trees by the Principle of Math Induction.
5(a) A minimum salesman walk is a closed walk of shortest possible length which includes all the vertices of G.

(b) Let H be the graph obtained by adding a new vertex $v_{p+1}$ to G and edges from $v_{p+1}$ to each of the vertices of G. Then H has $p+1$ vertices and for any pair of non-adjacent vertices x & y in H, we have
\[ \deg_H(x) + \deg_H(y) = [\deg_G(x) + 1] + [\deg_G(y) + 1] \geq (p-1) + 2 = p+1. \]
So by Ore's theorem, H has a Hamilton cycle.

Now if we delete the vertex $v_{p+1}$ from H, we will get a Hamilton path P of G. Thus G has a Hamilton path.

6(a) A simple polyhedron is a solid figure which is bounded by plane polygonal faces and which can be continuously distorted into a solid sphere.

(b) Let $A_1, \ldots, A_r$ be the regions of a planar embedding of the polyhedron G. Since G has no triangular faces, $e(A_i) = 4$ for each $A_i$. So
\[ 4r = 4 + 4 + \cdots + 4 \text{ (r times)} \leq e(A_1) + e(A_2) + \cdots + e(A_r) = 2q \text{ because each edge is in 2 regions.} \]
\[ 2r \leq q. \]
Since G is a simple polyhedron, the embedding will be a connected planar graph. So by Euler's formula $r = q + 2 - p$. Thus
\[ 2(q + 2 - p) \leq q \Rightarrow 2q + 4 - 2p \leq q \Rightarrow q \leq 2p - 4. \]
Thus $q(G) \leq 2p(G) - 4$ and we are done.