Chapter 6
Planarity

Section 6.1 Euler's Formula

In Chapter 1 we introduced the puzzle of the three houses and the three utilities. The problem was to determine if we could connect each of the three utilities with each of the three houses so that none of the utility lines crossed. We attempted a drawing of the graph model $K_{3,3}$ in Figure 1.1.2. In this chapter we will show that no such drawing is possible in the plane.

A (p, q) graph G is said to be *embeddable in the plane* or *planar* if it is possible to draw G in the plane so that the edges of G intersect only at end vertices. If such a drawing has been done, we say that a *plane* embedding of the graph has been found.

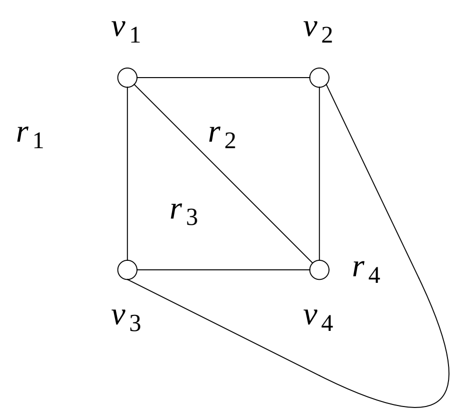


Figure 6.1.1. A plane embedding of K_4.

Given a plane graph G, a *region* of G is a maximal section of the plane for which any two points may be joined by a curve. Intuitively, a region is the connected section of the plane bounded (often enclosed) by some set of edges of G. In Figure 6.1.1, the regions of G are labeled r_1, r_2, r_3 and r_4. The region r_1 is bounded by the edges v_1v_2, v_2v_3 and v_3v_1, while r_2 is bounded by v_1v_2, v_2v_4 and v_4v_1. Further, note that every plane graph has a region similar to the region r_1. This region, which is actually not enclosed, is called the *exterior region.* It is also the case that given any planar graph G, there is an embedding of G with any region as the exterior region.

One of the most useful results for dealing with planar graphs relates the order, size and number of regions of the graph. This result is originally from Euler [1].

Theorem 6.1.1 If G is a connected plane (p, q) graph with r regions, then $p - q + r = 2$.

Proof. We proceed by induction on the size of the graph. If $q = 0$, then since G is connected, $p = 1$ and $r = 1$, and the result follows. Now, assume the result is true for all connected plane graphs with fewer than q edges $(q \geq 1)$ and suppose that G is a connected plane (p, q) graph. If G is a tree, then $p = q + 1$ and $r = 1$, and the result follows easily. If G is not a tree, then let e be an edge of G on some cycle C and consider $H = G - e$. Since e is not a bridge, H is clearly connected and planar. Further, H has p vertices, $q - 1$ edges and $r - 1$ regions; thus, by the induction hypothesis,

$$p - (q - 1) + (r - 1) = 2.$$

But then, inserting e back into H to form G results in one more edge and one more region, so G clearly satisfies the formula; that is,

$$p - q + r = 2. \quad \square$$

Since p, q and 2 are all fixed constants for a given planar graph G, it follows from Euler's formula that any two planar embeddings of a connected graph must have the same number of regions. Thus, we can simply refer to the number of regions of the planar graph without regard to the embedding.

It seems reasonable to think that we cannot simply continue to insert edges into a plane graph indefinitely while still maintaining its planarity. Thus, we would like to obtain an upper bound on the number of edges that a planar graph of given order can contain. If we continue to insert edges into a planar graph G, until, for every pair of nonadjacent vertices x and y, the graph $G + xy$ is nonplanar, we say the graph G is a *maximal planar graph.* But under what conditions can we insert an edge into G and have it remain planar? If we consider a plane embedding of G with a region r and if r is bounded by a cycle containing four or more edges, then we can certainly insert one of the missing edges between two vertices on this boundary. Further, there is no problem using this region for the embedding of the new edge. Thus, as long as we can find a region whose boundary contains four or more edges (or if the region is not enclosed, as in the case of a tree), we can continue to insert edges. For this reason, maximal planar graphs are sometimes called *triangulated planar graphs* or simply *triangulations* (see Figure 6.1.2). With this in mind, we can now develop a relationship between the order and size of maximal planar graphs.

Theorem 6.1.2 If G is a maximal planar (p, q) graph with $p \geq 3$, then

$$q = 3p - 6.$$

Proof. Let r be the number of regions of G. Note that the boundary of every region is a triangle and that each edge of G lies on the boundary of two such regions. Thus, if we

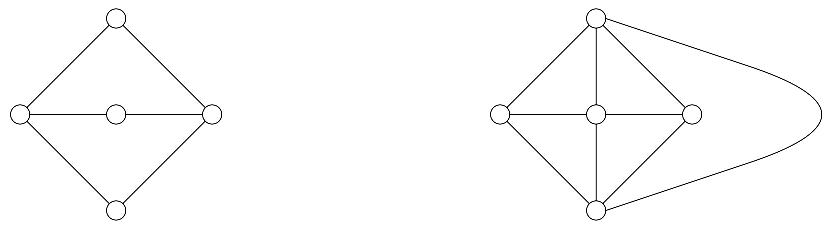


Figure 6.1.2. Inserting edges into $K_{2,3}$ to obtain a maximal planar graph.

sum the number of edges on the boundary of a region over all regions, we obtain $3r$. We note that this sum also counts each edge twice; thus, we obtain the relation $3r = 2q$. Now, applying Euler's formula, we see that

$$p - q + \frac{2q}{3} = 2$$

or

$$q = 3p - 6. \qquad \square$$

Corollary 6.1.1 If G is a planar (p, q) graph with $p \geq 3$, then

$$q \leq 3p - 6.$$

Proof. Merely add edges to G until it is maximal planar. The resulting graph has order p and size $q^* \geq q$. Further, from Theorem 6.1.2 we see that

$$q \leq q^* = 3p - 6,$$

and the result follows. $\square$

Another important consequence of Euler's formula is the following result.

Corollary 6.1.2 Every planar graph G contains a vertex of degree at most 5, that is, $\delta(G) \leq 5$.

Proof. Suppose $G = (V, E)$ is a planar (p, q) graph with $V = \{ x_1 , \ldots , x_p \}$. If $p \leq 6$, the result must hold, so suppose that $p \geq 7$. We know that $q \leq 3p - 6$; hence,

$$\sum_{i=1}^{p} deg\ x_i = 2q \leq 6p - 12.$$

But if all p vertices of G have degree 6 or more, then $2q \geq 6p$, a contradiction. Hence, G contains a vertex of degree 5 at most. $\square$

We are finally in a position to determine two very important nonplanar graphs. The first is the graph of our utilities problem, namely $K_{3,3}$, and the second is the complete graph K_5 (see Figure 6.1.3).

Corollary 6.1.3 The graph $K_{3,3}$ is nonplanar.

Proof. Suppose that $K_{3,3}$ is planar and let G be a plane embedding of $K_{3,3}$. Since $K_{3,3}$ has no triangles (in fact, no odd cycles at all), every region of G must contain at least four edges. Thus,

$$4r \leq 2q = 18.$$

But then, $r \leq 4$. Hence, by Euler's formula,

$$2 = p - q + r \leq 6 - 9 + 4 = 1,$$

a contradiction. □

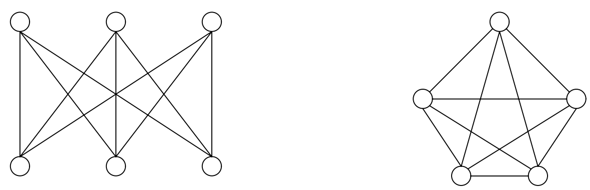


Figure 6.1.3. The nonplanar graphs $K_{3,3}$ and K_5.

Corollary 6.1.4 The graph K_5 is nonplanar.

Proof. If K_5 were planar, then by Corollary 6.1.1, K_5 would satisfy the bound $q \leq 3p - 6$. But then we would have that

$$10 = |E(K_5)| \leq 3(5) - 6 = 9,$$

again producing a contradiction. □

Section 6.2 Characterizations of Planar Graphs

There are actually a surprisingly large number of different characterizations of planar graphs. These range from topological in nature, to geometric, to structural. They entail techniques ranging from finding certain kinds of subgraphs to constructing alternate kinds of graphs and recognizing properties of these graphs. We shall consider three basic characterizations in this section and another in Chapter 9.

The oldest and most famous of all the characterizations of planar graphs is that of Kuratowski [5] and is topological in nature. The fundamental idea rests on the graphs $K_{3,3}$ and K_5 that we showed were nonplanar in the previous section. Kuratowski proved that these are the two "fundamental" nonplanar graphs. That is, he showed that any nonplanar graph must contain as a subgraph a graph that is closely related to at least one of these two graphs and that planar graphs do not contain such subgraphs. To understand what we mean by closely related requires a bit more terminology.

Recall that by a *subdivision* of an edge $e = xy$, we mean that the edge xy is removed from the graph and a new vertex w is inserted in the graph, along with the edges wx and wy. The essential idea behind Kuratowski's theorem is: If there is a problem drawing G in the plane because the edge xy will cross other edges, then there is still a problem drawing the path x, w, y (and conversely). Thus, in a sense, the original edge is a simpler structure to deal with, but the subdivided edge is just as large a problem as far as planarity is concerned.

We say a graph H is *homeomorphic from* G if either H is isomorphic to G or H is isomorphic to a graph obtained by subdividing some sequence of edges of G. We say G is *homeomorphic with* H if both G and H are homeomorphic from a graph F. The relation "homeomorphic with" is an equivalence relation on graphs (see the exercises). Hence, the set of graphs may be partitioned into classes; two graphs belong to the same class if, and only if, they are homeomorphic with each other. In Figure 6.2.1 we show a graph homeomorphic from $K_{3,3}$. Can you identify the vertices that correspond to the subdivided edges and the vertices that correspond to those of the original $K_{3,3}$?

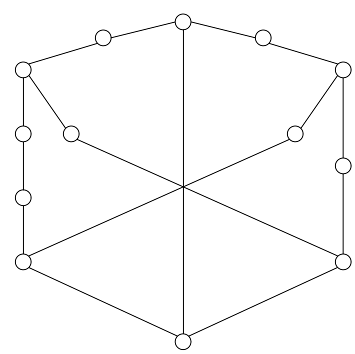


Figure 6.2.1. A subdivision of $K_{3,3}$.

We now present Kuratowski's characterization of planar graphs.

Theorem 6.2.1 (Kuratowski's Theorem [5]) A graph G is planar if, and only if, G contains no subgraph homeomorphic with K_5 or $K_{3,3}$.

Proof. Clearly a graph is planar if, and only if, each of its components is planar. Thus, we will restrict our attention to connected graphs. It is also straightforward to show that a graph is planar if, and only if, each of its blocks is planar (see exercise 4 in Chapter 6). Thus, we further restrict our attention to blocks other than K_2.

We know from our previous work that if a graph is planar then it does not contain a subgraph homeomorphic with K_5 or $K_{3,3}$. Thus, to complete the proof, it is sufficient to show that if a block contains no subgraph homeomorphic with K_5 or $K_{3,3}$, then it is planar. Suppose instead that this is not the case. Then among all nonplanar blocks not containing subgraphs homeomorphic with either K_5 or $K_{3,3}$, let G be one of minimum size.

We first claim that $\delta(G) \geq 3$. To prove this claim, note that since G is a block, it contains no vertices of degree 1. Suppose that G contained a vertex v with $deg\ v = 2$ and further suppose that v is adjacent with u and w. Consider two cases:

Case 1: Suppose uw is an edge of G. Then $G - v$ is still a block. Further, since $G - v$ is a subgraph of G, it follows that $G - v$ contains no subgraph homeomorphic with K_5 or $K_{3,3}$. But G is a nonplanar block of minimum size with this property; thus, $G - v$ must be planar. However, in any plane embedding of $G - v$, the vertex v and its incident edges vu and vw can be inserted so that the resulting graph (namely, G) is also plane. This contradicts the fact that G was nonplanar.

Case 2: Suppose that uw is not an edge of G. Then the graph $G_1 = G - v + uw$ is a block with size less than that of G. Furthermore, G_1 contains no subgraph F homeomorphic with either K_5 or $K_{3,3}$. Suppose instead that it did; then such a subgraph must contain the edge uw or G itself would have contained a subgraph homeomorphic with K_5 or $K_{3,3}$, which is not possible. But then $F - uw + vu + vw$ is a subgraph of G that is homeomorphic from F and, hence, implies that G contains a subgraph homeomorphic with either K_5 or $K_{3,3}$, again a contradiction.

In either case, we arrive at a contradiction, and, hence, $\delta(G) \geq 3$ as claimed.

Now, by exercise 32 in Chapter 2, we see that G is not a minimal block. Thus, there exists an edge $e = uv$ such that $H = G - e$ is still a block. Since H also has no subgraphs homeomorphic with K_5 or $K_{3,3}$ and has fewer edges than G, we see that H must be planar. Since H is a block other than K_2, Theorem 2.2.4 implies that H possesses cycles containing both u and v. Thus, suppose that H is a plane graph with a cycle C containing u and v and that C has been chosen with a maximum number of interior regions, say

$$C:\ u = v_0, v_1, \ldots, v_i = v, \ldots, v_n = u, \text{ where } 1 < i < n - 1.$$

For convenience, let the *interior subgraph* (*exterior subgraph*) of H be the subgraph of G induced by those edges lying in the interior (exterior) of C. With this in mind, we note the following observations about H.

Observation 1. Since G is nonplanar, both the interior and exterior subgraphs are nonempty; otherwise, the edge e could be inserted into H in the appropriate region to obtain a plane embedding of G.

Observation 2. No two vertices of the set $\{ v_0, v_1, \ldots, v_i \}$ are connected by a path in the exterior subgraph of H. If they were, this would contradict our choice of C as having the maximum number of interior regions. A similar statement applies to the set $\{ v_i, v_{i+1}, \ldots, v_n \}$. The observations about these two sets, along with the fact that $H + e$ is nonplanar, imply the existence of a $v_j - v_k$ path P in the exterior subgraph of H, where $0 < j < i < k < n$. Further, note that no vertex of P different from v_j and v_k is adjacent to a vertex of C other than v_j or v_k. If it were, we would again contradict our choice of C as having the maximum number of interior regions. Furthermore, any path joining a vertex of P with a vertex of C must contain at least one of v_j or v_k.

Let H_1 be the component of $H - \{ v_l \mid 0 \le l < n,\ l \ne j, k \}$ containing P. By our choice of C, we know that H_1 cannot be inserted in the interior of C in a plane manner. This fact, along with the assumption that G is nonplanar, implies that the interior of H must contain one of the following:

a. A $v_r - v_s$ path P_1, $0 < r < j$, $i < s < k$ or equivalently $j < r < i$ and $k < s < n$, none of whose vertices different from v_r and v_s belongs to C. In this case, we see that the graph contains a subgraph homeomorphic with $K_{3,3}$ with partite sets $\{ v_j, v_s, v_i \}$ and $\{ v_k, v_0, v_r \}$ (see Figure 6.2.2).

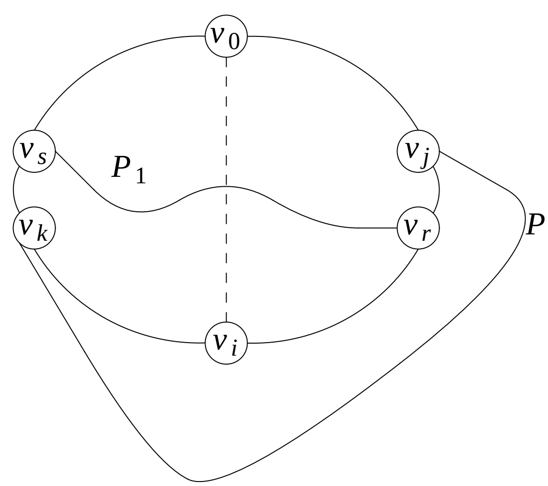


Figure 6.2.2. The case a.

b. A vertex $x \notin V(C)$ that is connected to C by three internally disjoint paths such that the end vertex of one such path P_1 is one of v_0, v_i, v_j and v_k. If P_1 ends at v_0, the end vertices of the other paths are v_r and v_s where $j \le r < i$ and $i < s \le k$, but not both $r = j$ and $s = k$, hold. If P_1 ends at any of v_i, v_j or v_k, we obtain three analogous cases. In this case we again find a subgraph homeomorphic with $K_{3,3}$ with partite sets $\{ x, v_k, v_i \}$ and $\{ v_0, v_s, v_r \}$ (see Figure 6.2.3).

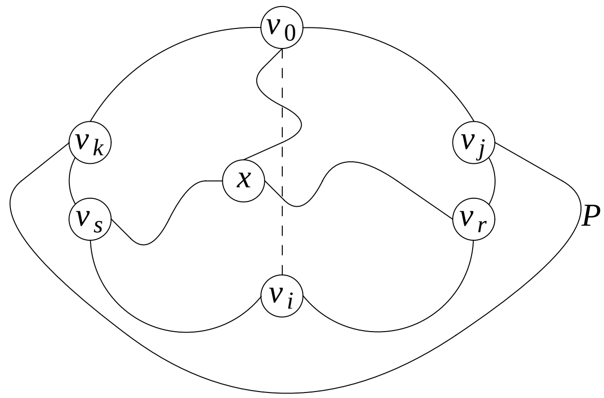


Figure 6.2.3. The case b.

c. A vertex $x \notin V(C)$ that is connected to C by three internally disjoint paths P_1, P_2 and P_3 where the end vertices of the paths other than x are three of v_0, v_i, v_j and v_k, say v_0, v_i and v_j, respectively, together with a $v_t - v_k$ path P_4 ($v_t \neq v_0, v_i, x$) where v_t is on P_1 or P_2, and P_4 is disjoint from P_1 and P_2, and C except for v_t and v_k. The remaining choices for P_1, P_2 and P_3 produce three analogous cases. In this case, we again find a subgraph homeomorphic with $K_{3,3}$ using the sets $\{v_0, v_t, v_j\}$ and $\{x, v_k, v_i\}$ (see Figure 6.2.4).

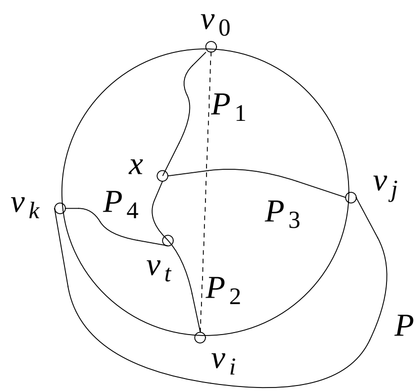


Figure 6.2.4. The case c.

d. A vertex $x \notin V(C)$ that is connected to vertices v_0, v_i, v_j and v_k by four internally disjoint paths. In this case a subgraph homeomorphic with K_5 is produced (see Figure 6.2.5).

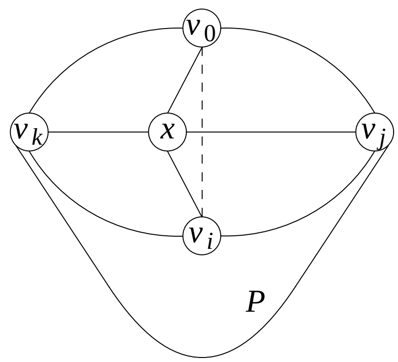


Figure 6.2.5. The case d.

These cases exhaust the possibilities, and each produces a subgraph homeomorphic with either K_5 or $K_{3,3}$. But our assumptions are contradicted; hence, no such graph G can exist and the proof is completed. □

Next, we consider a more constructive look at planar graphs centering on the cycles of the graph. If $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E)$, then a *piece* of G relative to G_1 is either an edge $e = uv$ where $e \notin E_1$ and $u, v \in V(G_1)$ or a connected component of $(G - G_1)$ plus any edges incident to vertices of this component (see Figures 6.2.6 and 6.2.7).

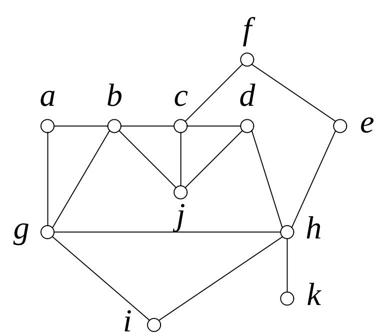


Figure 6.2.6. A plane graph G with initial cycle C: b, c, d, h, i, g, b.

For any piece P relative to G_1, the vertices of P in G_1 are called the *contact vertices of P*. If a piece has two or more contact vertices, it is called a *segment.* If C is a cycle of G, then C (when embedded) divides the plane into two regions, one interior to C, the other exterior. Two segments are *incompatible* if when placed in the same region of the plane determined by C, at least two of their edges cross. An example of this is shown in Figure 6.2.8.

Note that since pieces that are not segments have only one contact vertex, their embeddings are relatively easy. In order to manage the many potential segments of a graph, we introduce another graph whose structure depends on the cycle C. This new

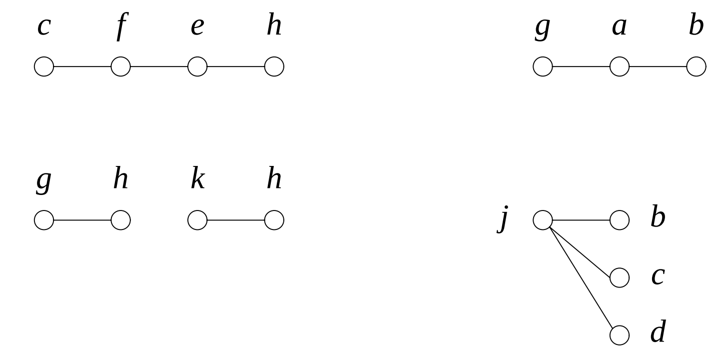


Figure 6.2.7. Pieces and segments of the plane graph G relative to C.

graph is called an *auxiliary graph* and is denoted $P(C)$. The graph $P(C)$ is constructed as follows: $P(C)$ has a vertex corresponding to each segment of the graph G relative to C, and an edge joins two such vertices if, and only if, the corresponding segments are incompatible. In attempting to embed the segments of G relative to C, we clearly must have incompatible segments in different regions. The compatible segments are independent vertices in $P(C)$. If $P(C)$ were actually a bipartite graph, then the segments represented by the vertices in one partite set could be embedded in the same region (either inside C or outside C) without conflict. Further, the segments represented by the vertices of the other set could be embedded in the second region without conflict and without affecting the segments in the first region. That this is both a necessary and sufficient condition is shown in our next result (Demoucron, Malgrange and Pertuiset [2]).

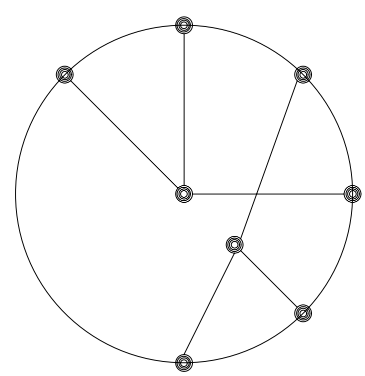


Figure 6.2.8. Incompatible segments.

Theorem 6.2.2 The graph G is planar if, and only if, for every cycle C of G, the auxiliary graph $P(C)$ is bipartite.

Proof. Suppose G is not planar. Then by Kuratowski's theorem, G contains a subgraph homeomorphic with either K_5 or $K_{3,3}$. Suppose G contains either K_5 or $K_{3,3}$ (the generalization to graphs homeomorphic with these is similar). In either case, we can select a cycle C such that $P(C)$ is not bipartite. For K_5 the segments are two edges and a vertex and its incident edges (see Figure 6.2.9 for example), while in $K_{3,3}$, the segments are all single edges (Figure 6.2.10). In either situation, any two of the three segments are incompatible. Thus, $P(C)$ is not bipartite.

Conversely, if G is planar, the fact that $P(C)$ is bipartite follows from our earlier discussion. □

Example 6.2.1. Consider K_5 with $V(K_5) = \{1, 2, 3, 4, 5\}$ and let C be the cycle 1, 2, 3, 4, 1. Then C and the segments of $K_5 - C$ are shown in Figure 6.2.9. It is easy to see that any two segments are incompatible; thus, $P(C) = K_3$, which is clearly not bipartite. Thus, we have another proof that K_5 is nonplanar. A similar argument applies to $K_{3,3}$ (see Figure 6.2.10). □

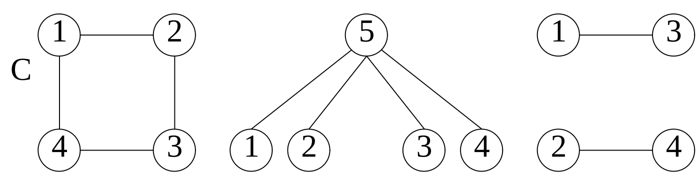


Figure 6.2.9. The cycle C and its relative segments in K_5.

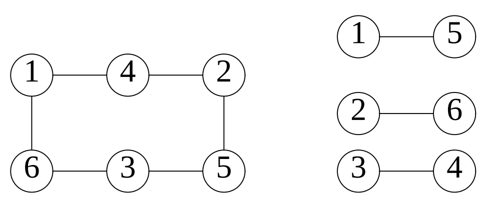


Figure 6.2.10. The cycle C and its relative segments in $K_{3,3}$.

Another characterization of planar graphs is from Whitney [6]. This characterization is of a more geometric nature. Given a plane graph G, we construct a pseudograph G^* as follows: For each region of G, we associate a vertex of G^*. Two vertices of G^* are joined by an edge corresponding to each edge that belongs to the boundary of both corresponding regions of G. Further, a loop is added at any vertex of G^* for each bridge of G that belongs to the boundary of the corresponding region. Note that each edge of G^* can be drawn in such a way that it crosses the associated edge of G (but no other edge of G). The pseudograph G^* is called the (*geometric*) *dual of G* (see Figure 6.2.11). It is clear from our discussion that G^* is planar; however, it may not be unique. Rather, it depends on the embedding selected for G.

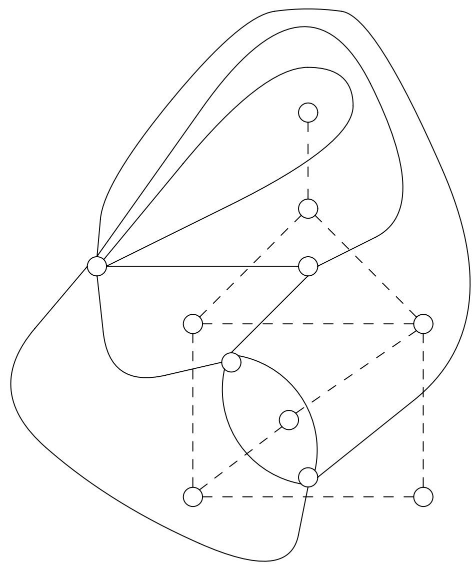


Figure 6.2.11. A graph (dashed lines) and its geometric dual (solid lines).

We require a slightly stronger idea in order to make use of duals in characterizing planar graphs. This idea is combinatorial in nature. Let G and $\tilde{G}$ be two graphs with a $1-1$ correspondence between their edges and let C be any cycle in G. The graph $\tilde{G}$ is called the *combinatorial dual of* G if, and only if, the set of edges $\tilde{C}$ in $\tilde{G}$ corresponding to the edges of C in G form an edge cut set. It is important to note that this definition makes no reference to planar graphs or the possible embedding. We now wish to investigate the relationship between the two duals we have introduced.

Theorem 6.2.3 Every planar graph G has a planar combinatorial dual.

Proof. We already know that every planar graph has a planar geometric dual. Hence, given an embedding of G, construct the geometric dual G^* as before. Now, given any cycle of G, this cycle must divide the plane into two regions. Thus, we can partition the vertices of G^* into two nonempty subsets A and B, according to their locations inside C or outside C. Removal of the set of edges C^* which correspond in G^* to the edges of C clearly separates A and B. Thus, C^* is an edge cut set of G^*.

Given any edge cut set C^* of G^*, we would like to determine the corresponding set of edges C in G and show that C forms a cycle in G. Suppose this is not the case. By our construction, only one vertex of G^* lies in any region determined by the plane embedding of G. Consider the edges incident to some vertex v of G. Each of these edges lies on the boundary of two regions of G in this embedding, and each of these regions contains one vertex of G^*. (Of course, if the edge is a bridge, the two regions are equal.) The end vertices of the corresponding edges of G^* are then determined, and these edges bound a region of G^*. Hence, every vertex of G lies in one region of G^*. If C is not a cycle of G, then there are at least two edges of C whose end vertices are not adjacent to other vertices of C. If C^* was a cut set, then these end vertices would have to lie in the same region of G^*. But this is a contradiction, since each region contains only one vertex of G. □

What we have shown is that if G_1 is planar, then the combinatorial dual actually coincides with the geometric dual. Our construction makes the following corollary immediate.

Corollary 6.2.1 If G is a graph with geometric dual G^*, then $(G^*)^* = G$.

We can now show that the existence of a combinatorial dual characterizes planar graphs.

Theorem 6.2.4 A graph G is planar if, and only if, G has a combinatorial dual.

Proof. From Theorem 6.2.3, we know that every planar graph has a combinatorial dual. Thus, we need only show that every nonplanar graph has no combinatorial dual.

From our definition of combinatorial dual, we see that G has a combinatorial dual if, and only if, each of its subgraphs has a combinatorial dual. Also, if a graph has a combinatorial dual, then so does any graph homeomorphic from it. Further, G is nonplanar if, and only if, it contains a subgraph homeomorphic with K_5 or $K_{3,3}$. Thus, it suffices to show that these graphs have no combinatorial dual.

To see that K_5 has no combinatorial dual, suppose instead that it did, say $\tilde{K}_5$. Since K_5 has ten edges and is an ordinary graph with edge connectivity 4, we observe that there are no $2-$cycles and no edge cut sets with two or three edges. In fact, K_5 does have edge cut sets with four or six edges. These properties imply that $\tilde{K}_5$ has ten edges, no vertices of degree less than 3 and no 2-cycles, but it does contain $4-$ and $6-$cycles. However, these properties are mutually incompatible.

To see that $K_{3,3}$ has no combinatorial dual, we again suppose that it did have such a dual, say $\tilde{K}_{3,3}$. Clearly, $K_{3,3}$ has no 2-cycles and no odd cycles, but it certainly contains $4-$ and $6-$cycles. Thus, $\tilde{K}_{3,3}$ has no edge cut set with fewer than four edges. Also, the degree of every vertex of $\tilde{K}_{3,3}$ is at least 4. Thus, there are at least twelve edges in $\tilde{K}_{3,3}$. But there are only nine edges in $K_{3,3}$, and, hence, by our construction there can be only nine edges in $\tilde{K}_{3,3}$, producing the desired contradiction. □

Section 6.3 A Planarity Algorithm

In the next two sections, we will discuss two well-known planarity algorithms. The first, from Demoucron, Malgrange, and Pertuiset [2], is somewhat easier to understand and uses the ideas of segments. In the next section, we will discuss a linear planarity algorithm.

In any attempt at an algorithm to determine if a graph is planar, there are several preliminary tests that can really help simplify the process. These include:

1. If $|E| > 3p - 6$, then the graph must be nonplanar.
2. If the graph is disconnected, consider each component separately.
3. If the graph contains a cut vertex, then it is clearly planar if, and only if, each of its blocks is planar. Thus, we can limit our attention to $2-$connected graphs.
4. Loops and multiple edges change nothing; hence, we need only consider graphs.
5. A vertex of degree 2 can certainly be replaced by an edge joining its neighbors. This *contraction* of all vertices of degree 2 constructs a homeomorphic graph with the smallest number of vertices. Certainly, a graph is planar if, and only if, the contraction is planar.

Thus, if we do some preprocessing, our task will sometimes be greatly reduced. Certainly tests 4 and 5 have the greatest effect on the graph under consideration, while test 1 can produce the most rapid benefit.

The key concept in the Demoucron, Malgrange and Pertuiset algorithm is the following idea: Let $\hat{H}$ be a plane embedding of the subgraph H of G. If there exists a planar embedding of G (say $\hat{G}$) such that $\hat{H} \subseteq \hat{G}$, then $\hat{H}$ is said to be *G-admissible.* As an example, consider the graph $G = K_5 - e_1$ where $V = \{1, 2, 3, 4, 5\}$ and the missing edge e_1 is from vertex 1 to vertex 5. The graphs of Figure 6.3.1 show a G-admissible and a G-inadmissible embedding of the subgraph $H = G - e_2$ where e_2 is the edge from vertex 1 to vertex 2.

Let S be any segment of G relative to a subgraph H. Then S can be drawn in a region r of $\hat{H}$, provided all the contact vertices of S lie in the boundary of r. This allows us to extend the embedding of $\hat{H}$ to include at least part of S. The strategy of the algorithm is to find a sequence of subgraphs $\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_{|E|-p+2} = G$ (do you know why there are this many potential subgraphs?) such that $\hat{H}_i \subset \hat{H}_{i+1}$ and such that $\hat{H}_i$ is G-admissible (if possible). In this way, a planar embedding of G (if one exists) can be constructed, or we will discover some segment S which cannot be compatibly embedded in any region.

The algorithm begins by finding any cycle in G and embedding it. This cycle is the first subgraph, $\hat{H}_1$. The algorithm then attempts to find the set of segments relative to the

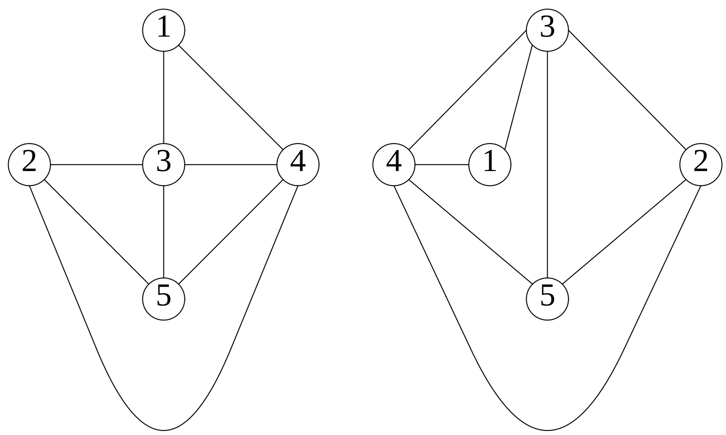


Figure 6.3.1. An admissible and inadmissible embedding of *H*.

current $\hat{H}_i$. For each such segment *S*, the set $R(S,\ \hat{H}_i)$ is found, where $R(S,\ \hat{H}_i)$ is the set of regions in which *S* can be compatibly embedded in $\hat{H}_i$. If there exists a segment *S* which has only one such region *r*, then $\hat{H}_{i+1}$ is constructed by drawing a path *P* between two of the contact vertices of *S* in the region *r*. If no such segment exists, then a path *P* between two contact vertices of any segment is constructed. In either case, the path *P* divides the region in which it is embedded into two regions. The process is then repeated if necessary. We now state the Demoucron, Malgrange and Pertuiset (DMP) algorithm [2].

Algorithm 6.3.1 DMP Planarity Algorithm.
Input: A preprocessed block (after applying tests $1-5$).
Output: The fact that the graph is planar or nonplanar.
Method: Look for a sequence of admissible embeddings beginning with some cycle *C*.

1. Find a cycle *C* and a planar embedding of *C* as the first subgraph $\hat{H}_1$. Set $i \leftarrow 1$ and $r \leftarrow 2$.

2. If $r = |E| - p + 2$,
 then stop;
 else determine all segments *S* of $\hat{H}_i$ in *G*, and for each segment *S* determine $R(S,\ \hat{H}_i)$.

3. If there exists a segment *S* with $R(S,\ \hat{H}_i) = \varnothing$,
 then stop and say *G* is nonplanar;
 else if there exists a segment *S* such that $|R(S,\ \hat{H}_i)| = 1$,
 then let $\{\ R\ \} = R(S,\ \hat{H}_i)$;
 else let *S* be any segment and *R* any region in $R(S,\ \hat{H}_i)$.

4. Choose a path P in S connecting two contact vertices. Set $H_{i+1} = H_i \cup P$ to obtain the planar embedding $\hat{H}_{i+1}$ with P placed in R.
5. Set $i \leftarrow i + 1$, $r \leftarrow r + 1$ and go to step 2.

Example 6.3.1. We now determine an embedding of $G = K_5 - \{ e_1, e_2 \}$, using Algorithm 6.3.1, where $V = \{ 1, 2, 3, 4, 5 \}$ and where e_1 is the edge from vertex 4 to vertex 5 and e_2 is the edge from vertex 2 to vertex 4.

In step 1 we determine any cycle $C = H_1$ of G, for example; the cycle $1, 2, 3, 4, 1$, and we embed it in the plane as shown in Figure 6.3.2 as $\hat{H}_1$. We also set $i = 1$ and $r = 2$.

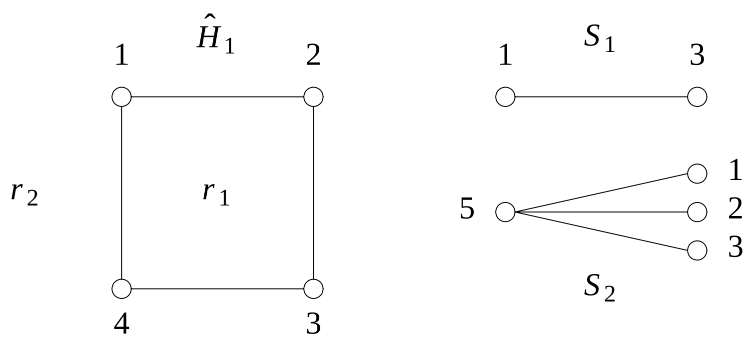


Figure 6.3.2. The first graph, $\hat{H}_1$, and its segments.

In step 2 we determine the segments relative to $\hat{H}_1$. These consist of the edge from 1 to 3 and the vertex 5 and its incident edges 1 to 5, 2 to 5 and 3 to 5. Call these segments S_1 and S_2, respectively. Then

$$R(S_1, \hat{H}_1) = \{ r_1, r_2 \} = R(S_2, \hat{H}_1).$$

Since both sets contain more than one region, we can select either segment for the embedding. Suppose we select S_2. Next, we must choose a path between two contact vertices, say this path P is $1, 5, 2$, and we embed it in any region in $R(S_2, \hat{H}_1)$. Without loss of generality, suppose it is embedded in the exterior region r_2 of $\hat{H}_1$. We obtain the graph $\hat{H}_2$ shown in Figure 6.3.3. This clearly creates a third region r_3, and the regions are now as shown in Figure 6.3.3. Next, update $r \leftarrow 3$ and $i \leftarrow 2$ and go to step 2. Again, select the segments relative to $\hat{H}_2$: the edge from 1 to 3 and the edge from 3 to 5. In step 3, since the edge from 3 to 5 can only be placed in r_2, it is the segment selected and embedded as shown in Figure 6.3.4, forming $\hat{H}_3$.

Finally, the only remaining segment is the edge from 1 to 3, and it can be drawn in region r_1 or r_2. The choice of region determines the final embedding. We select region r_1, and in this case we obtain the embedding $\hat{H}_4$, shown in Figure 6.3.5.

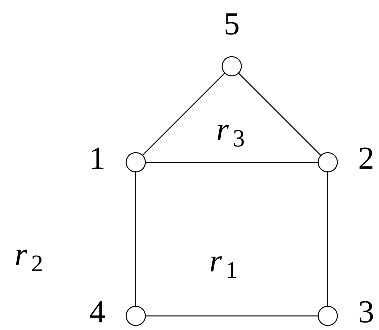


Figure 6.3.3. The graph $\hat{H}_2$ formed by inserting P in the exterior region.

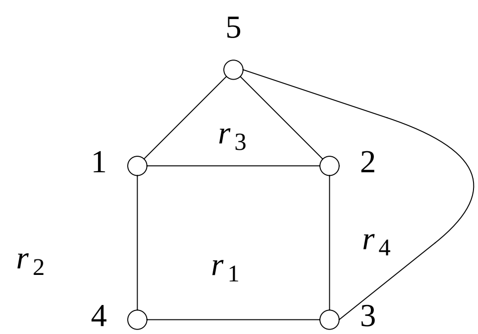


Figure 6.3.4. The graph $\hat{H}_3$ formed by inserting the edge from 3 to 5 in r_2.

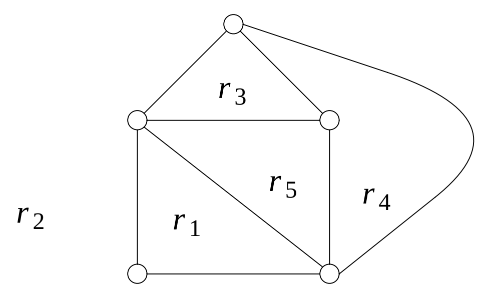


Figure 6.3.5. The graph $\hat{H}_4 = K_5 - \{ e_1 , e_2 \}$.

Section 6.4 The Hopcroft-Tarjan Planarity Algorithm

In this section we examine the most effective planarity algorithm (Hopcroft and Tarjan [4]). This algorithm has complexity $O(|V|)$ and is therefore a very effective method of testing for planarity. We will not be able to prove all aspects of the algorithm here, but rather we will try to establish enough for the reader to have a general understanding of the algorithm without necessarily having enough details to immediately begin programming it. For more details on the data structures necessary to program the algorithm, see [4]. To study the algorithm, we will break it into a number of separate sections and try to describe each of these sections individually.

The fundamental idea of the Hopcroft-Tarjan planarity algorithm is the typical one for planarity testing: Break the graph into a cycle and paths, and then fit these paths together in such a way as to find the embedding or the fact that the graph is not planar. We seek a cycle C and a collection of paths P_i so that when we embed C in the plane, the paths P_i can be embedded either inside C or outside C. If this can be done without interference, then G is planar. The difference in this algorithm is the strategy for finding the paths. In order to find the initial cycle and the subsequent paths, we will first perform a depth-first search on G and then, using the information obtained from this search, impose an ordering of the edges in each of the adjacency lists for the vertices of G. This will allow us to select paths with special properties that will facilitate the testing for planarity.

We begin by reordering the vertices and their adjacency lists. This reordering will make it possible for us to actually produce the paths easily and in such a way that paths from the same segment are generated consecutively. To do this, we perform a depth-first search on G, viewing the result as a digraph D_G, with its tree arcs and back arcs. From this point on we identify vertices by their number value $num(v)$, assigned by the DFS. We also assign each vertex a lowpoint value $lowpt(v)$, defined to be the number of the lowest vertex reachable from v or any of its descendents in the DFS tree using at most one back edge. When it is not possible to do this, then $lowpt(v) = v$. In a similar fashion, let $nextlopt(v)$ be the next lowest vertex below v, excluding $lowpt(v)$ reachable in the same manner. If there is no such vertex, then set $nextlopt(v) = v$. More formally, then, let S_v be the set of all vertices reachable from v using zero or more tree arcs and at most one back arc. Then we define

$$lowpt(v) = \min\ (S_v)$$

and

$$nextlopt(v) = \min\ (\{v\} \cup (S_v - lowpt(v)).$$

We now restate a result fundamental to the use of these functions and previous examined in Lemmas 2.2.3 and 2.2.4. Recall that it also supplies us with a means of finding cut vertices.

Theorem 6.4.1 If $G = (V, E)$ is a connected graph with DFS search tree T and back edges B, then $u \in V$ is a cut vertex if, and only if, there exist vertices $v, w \in V$ such that $u \to v \in T$ and w is not a descendent of v in T and $lowpt(v) \geq num(u)$.

The last result implies that $num(v) \geq nextlopt(v) > lowpt(v)$, except when v is the root of the DFS tree T, in which case

$$lowpt(v) = nextlopt(v) = num(v) = 1.$$

Next, we reorder the adjacency lists of D_G so that during a DFS on D_G, the paths we desire will be generated in a useful order. In order to accomplish this goal, we need another function defined on $E(G)$. Let $e = v \to w \in E(D_G)$ and define

$$\phi(e) = \begin{cases} 2\,w & \text{if } e \text{ is a back edge} \\ 2\,lowpt(w) & \text{if } e \text{ is a tree edge and } nextlopt(w) \geq v \\ 2\,lowpt(w) + 1 & \text{if } e \text{ is a tree edge and } nextlopt(w) < v. \end{cases}$$

Now, we sort all arcs $v \to w$ incident to v into nondecreasing order based on their ϕ values and use this ordering in the adjacency list of each vertex. These new adjacency lists are called $adj(v)$ for each vertex v. The important point to notice here is that in $adj(v)$, a back edge going to a vertex of lower DFS number always precedes a back edge going to a vertex of higher DFS number and tree edges generally appear in nondecreasing order of their abilities to lead to a vertex below v using a single back edge. To illustrate what we have done so far, consider the graph of Figure 6.4.1.

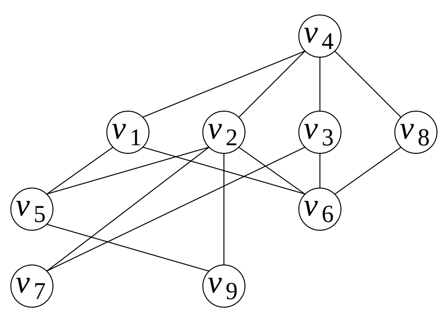


Figure 6.4.1. A graph G to test for planarity.

The digraph of Figure 6.4.2 shows D_G obtained from a DFS beginning at v_1. The search tree is shown with dashed arcs, and the back arcs are solid. The numbering alongside the vertices indicates the DFS number. Table 6.4.1 lists each vertex, its DFS number, *lowpt* value and *nextlopt* value.

Table 6.4.2 lists the arcs and their ϕ values. Now we can order the adjacency lists. This is shown in Table 6.4.3. Now that we have properly prepared the graph and its adjacency

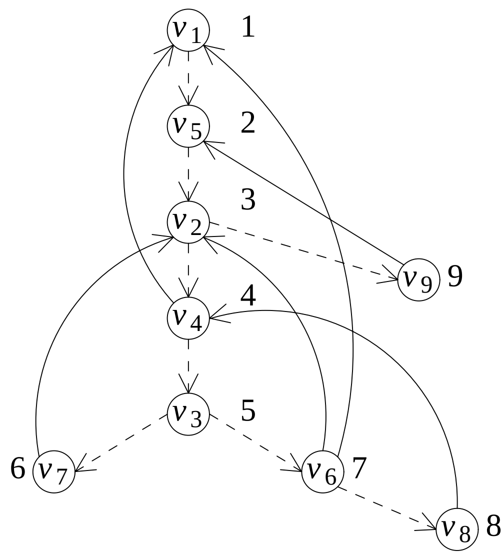


Figure 6.4.2. The DFS tree in dashed edges; back edges are solid.

vertex v	$num(v)$	$lowpt(v)$	$nextlopt(v)$
v_1	1	1	2
v_5	2	1	2
v_2	3	1	2
v_4	4	1	3
v_3	5	1	3
v_7	6	3	6
v_6	7	1	3
v_8	8	4	8
v_9	9	2	9

Table 6.4.1 Table of DFS numbers and function values.

lists, we are ready to use the lists to help us select both the first cycle and the subsequent paths. This decomposition, performed on D_G, will provide the pieces necessary to test for planarity. So, starting at vertex 1 (from now on we identify the vertex with its DFS number), follow tree edges until we reach a vertex z with the property that the first adjacency in $adj(z)$ is a back edge. We shall see that this back edge must go to vertex 1, thus forming the cycle we seek. Call the cycle C. In our example, $z = 4$, and the cycle $P_0 = C$ is $1, 2, 3, 4, 1$.

Now, starting at $z = 4$, follow the next adjacency indicated by $adj(z)$. Continue following tree edges until the first adjacency on a list forces us to traverse a back edge. This completes the first path of the decomposition. In our example this is the path

arc	ϕ	arc	ϕ	arc	ϕ
$v_1 \rightarrow v_5$	2	$v_5 \rightarrow v_2$	2	$v_2 \rightarrow v_4$	2
$v_3 \rightarrow v_7$	6	$v_3 \rightarrow v_6$	3	$v_6 \rightarrow v_8$	8
$v_7 \rightarrow v_2$	6	$v_8 \rightarrow v_4$	8	$v_6 \rightarrow v_2$	6
$v_2 \rightarrow v_9$	4	$v_9 \rightarrow v_5$	4	$v_4 \rightarrow v_3$	3
$v_4 \rightarrow v_1$	2	$v_6 \rightarrow v_1$	2		

Table 6.4.2 Table of arcs and ϕ values.

vertex v	$adj(v)$	vertex v	$adj(v)$
v_1	v_5	v_2	$v_4 \quad v_9$
v_3	$v_6 \quad v_7$	v_4	$v_1 \quad v_3$
v_5	v_2	v_6	$v_1 \quad v_2 \quad v_8$
v_7	v_2	v_8	v_4
v_9	v_5		

Table 6.4.3 The reordered adjacency lists.

P_1: 4, 5, 7, 1.

A new path is now started at the initial vertex of the back arc that ended the previous path. If this vertex has no unused arcs, we back up to the predecessor of this vertex on the present path. This process is continued until we have used all the arcs in D_G. In our example we find P_2: 7, 3, then P_3: 7, 8, 4, followed by P_4: 5, 6, 3 and P_5: 3, 9, 2. We now state the path decomposition algorithm.

Algorithm 6.4.1 Path Decomposition.
Input: The DFS numbered digraph D_G along with tree and back edges and ordered adjacencies.
Output: The path decomposition $P_0 = C, P_1, \ldots, P_k$.
Method: Use the ordered adjacencies from a base vertex s to find the paths.

1. Set $i \leftarrow 0$, $P_i \leftarrow \phi$.
2. Call PATH(1).

Procedure PATH(v)

1. For each $w \in adj(v)$ do the following:
2. Set $P_i \leftarrow P_i \cup \{ v \rightarrow w \}$.

3. If $v < w$,
   then PATH(w);
   else $i \leftarrow i + 1$ and $P_i \leftarrow \phi$.

The path decomposition obtained using this algorithm has many special properties ideal for use in the planarity test. We state some of these properties below; their proofs are left to the exercises.

1. The number of paths in the decomposition is $|E| - |V|$.
2. Every path P_i has only its end vertices in common with the union of the previously generated paths.
3. Algorithm 6.4.1 always selects the unused back edge to the lowest numbered vertex.
4. Let P_i and P_j be two paths with initial and terminal vertices s_i, t_i and s_j, t_j, respectively. If s_i is an ancestor of s_j (or possibly equal), then $t_i \le t_j$.
5. Let P_i and P_j be two paths with the same initial and terminal vertices s and t. Let v_i be the second vertex on P_i and v_j be the second vertex on P_j. If $s \to x_i$ is not a back edge and $nextlopt(x_i) < s$, then $s \to x_j$ is not a back edge and $nextlopt(x_j) < s$.

It is the final property that forces us to use the *nextlopt* function to break ties between two tree edges that start from the same vertex and have equal *lowpt* values for their terminal vertices.

This algorithm also uses the idea of a segment, but in a special way. Here, a *segment* of D_G with respect to C is either a single back edge with both vertices on C or the subgraph consisting of a tree edge $v \to w$ with $v \in C$ and $w \notin C$ and the directed subtree in T rooted at w along with all the back edges from this subtree. The vertex v of C at which a segment originates is called a *base vertex*. Algorithm 6.4.1 generates segments in decreasing order of their base vertices. All paths belonging to the same segment must be embedded together in the same region with respect to C. This is the motivation for trying to group these paths together in the generation process.

In our example graph, we find two segments. The first is formed from the paths P_1, P_2, P_3 and P_4, while the second is just the path P_5.

Now, we want to embed the segments. Initially, C divides the plane into two regions, one inside C and the other outside. We say the segment S is embedded *inside* C if the order of the edges encountered in a clockwise sweep around the base vertex v_i is $v_{i-1} \to v_i$, $v_i \to w$, $v_i \to v_{i+1}$. We say the segment is embedded *outside* C if the order is $v_{i-1} \to v_i$, $v_i \to v_{i+1}$, $v_i \to w$.

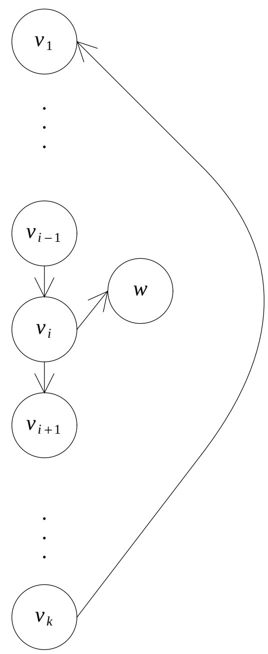


Figure 6.4.3. An embedding inside C.

After embedding C, we wish to embed the segments one by one in the order in which they are generated by Algorithm 6.4.1. In order to embed a segment, we consider the paths that compose the segment, and embed them one by one. By examining previously embedded paths, we can determine if the next path can be embedded in the region. If it cannot, then all the previously embedded segments that are blocking our embedding are moved to the other region. (Moving a segment to the other region may force the movement of even more segments.) If we still cannot embed the path after this rearrangement of segments, then we conclude that G is not planar. If, however, the path can be embedded, then we do so and continue to try to embed the rest of the segment, applying the embedding algorithm recursively. If we are successful in embedding this segment, we move on to the next segment.

The key to the embedding is the ability to easily test whether the first path of a segment can be embedded on a specific side of C. This can be done with the use of the following result.

Theorem 6.4.2 Let P be the first path in the current segment S and let P have base vertex v_i and terminal vertex v_j. If all segments generated prior to S have already been embedded, then P can be embedded in a region of C if there is no previously embedded back edge $x \to v_t$ in this region also entering C between vertices v_i and v_j. That is, no back edge $x \to v_t$ satisfies $v_j < v_t < v_i$. Furthermore, if there is such a back edge, then S cannot be embedded in this region of C.

This result makes it clear that the initial and terminal vertices of a path are all we really need to know when testing whether a given path can be embedded. (This is really the idea we mentioned earlier in connection with Kuratowski's theorem: An edge and a path are just as difficult to embed.) This easy criterion enhances the use of this algorithm.

So far we have talked about embedding the first path of the segment S. To determine what to do with the rest of the segment, we note that it can be embedded properly if, and only if, $S \cup C$ is planar. To determine the planarity of $S \cup C$, we apply the embeddability criterion recursively, until all paths in S are embedded in the plane or until we learn that we cannot embed G.

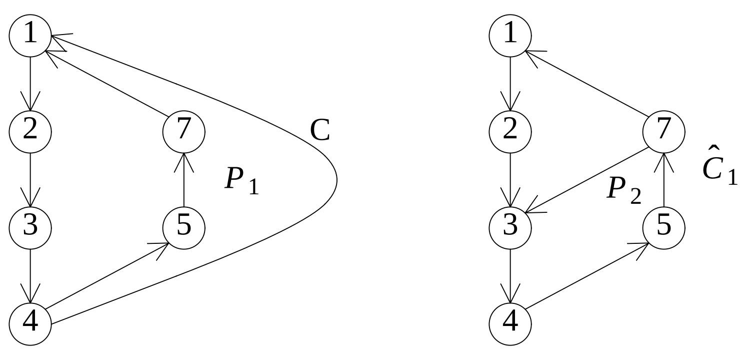


Figure 6.4.4. The first two paths embedded inside their respective cycles.

To perform the recursion on successive paths, we proceed as follows. If the path P has just been embedded, then P and the tree edges from the final vertex of P to the initial vertex of P form the new cycle $\hat{C}$. Removing $\hat{C}$ from $\hat{G} = S \cup C$ may further partition the remaining digraph into segments. These segments are again handled recursively. The first few cycles and embeddings of our example graph are easy (see Figures 6.4.4 and 6.4.5).

However, in attempting to embed P_4 inside $\hat{C}_3$, we find a conflict. Further, a conflict also exists outside of $\hat{C}_3$. But we cannot yet conclude that the graph is nonplanar. Working recursively, we now move the last segment embedded (one level earlier), outside of its cycle. Thus, P_3 is moved outside $\hat{C}_2$. The path P_4 still fails inside $\hat{C}_3$, but it can be properly embedded outside $\hat{C}_3$, as shown in Figure 6.4.6. The final segment is also easily embedded.

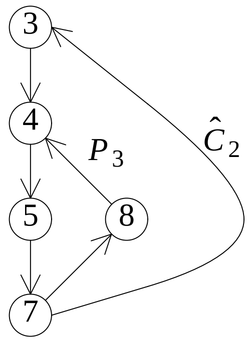


Figure 6.4.5. The third path is embedded inside $\hat{C}_2$.

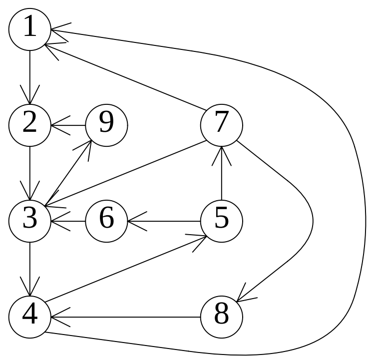


Figure 6.4.6. The final embedding.

We now present the Hopcroft-Tarjan algorithm.

Algorithm 6.4.2 The Hopcroft-Tarjan Planarity Algorithm.
Input: A 2−connected graph with $|E| \leq 3|V| - 6$.
Output: A determination of the planarity of the graph.
Method: Recursive testing of the paths from the path decomposition. They are embedded inside or outside the initial cycle and moved if necessary.

Procedure Planar(G)

1. Perform a DFS on G to obtain D_G.
2. Find a cycle C in D_G.
3. Construct a planar representation for C.
4. For each segment S formed in $G - C$, do the following:
   Apply the algorithm recursively to determine if $S \cup C$ is planar. That is, call

Planar($S \cup C$).
If $S \cup C$ is planar and S may be added to the planar representation already constructed, then add it; otherwise, halt and state that G is nonplanar.

To determine the complexity of the Hopcroft-Tarjan algorithm, note that the path-finding algorithm requires $O(|V| + |E|)$ operations. The embedding consists of the end vertices of the $|E| - |V|$ paths generated off the cycle. The embedding is done completely with stack operations (storing end vertices of paths), and pushing and popping entries takes constant time per entry. Since the total number of entries in any stack is $O(|V| + |E|)$, then we see that the entire algorithm takes only $O(|V| + |E|)$ operations. However, since we have restricted use of the algorithm to preprocessed graphs (recall preprocessing steps $1-5$ of the last section) satisfying the restriction that $|E| \le 3|V| - 6$, we see that the algorithm has complexity $O(|V|)$.

Section 6.5 Hamiltonian Planar Graphs

In Chapter 5 we studied many sufficient conditions implying that a graph was hamiltonian. However, until now we have said very little about necessary conditions for hamiltonian graphs. In this section we will show that such conditions do exist and that when we include planarity as an added condition, a very nice theorem (Grinberg [3]) results.

In order to present this result, we need a bit more terminology. Let G be a hamiltonian plane graph of order p. Further, let C be a fixed hamiltonian cycle in G. We say that an edge e is a *diagonal* with respect to C if e is not on C. Let r_i denote the number of regions of G containing exactly i edges that lie interior to C and let $\hat{r}_i$ denote the number of regions containing exactly i edges lying exterior to C. Then, using this notation, we can present Grinberg's theorem [3].

Theorem 6.5.1 Let G be a plane graph of order p with a hamiltonian cycle C. Then, with respect to this cycle,

$$\sum_{i=3}^{p} (i-2)(r_i - \hat{r}_i) = 0.$$

Proof. Consider a diagonal on the interior of C. This diagonal divides the region in which it is embedded into two regions. Therefore, if there are m diagonals in the interior of C, then

$$\sum_{i=3}^{p} r_i = m + 1.$$

But this implies that

$$m = \left(\sum_{i=3}^{p} r_i \right) - 1.$$

Let N denote the sum over all $m + 1$ interior regions of the number of edges bounding these regions. Then

$$N = \sum_{i=3}^{p} i r_i .$$

However, N also counts each interior diagonal twice and each edge of C once. Thus,

$$N = 2m + p.$$

Substituting for N and m in our last equation, we obtain

$$\sum_{i=3}^{p} i r_i = 2 \sum_{i=3}^{p} r_i - 2 + p.$$

Thus,

$$\sum_{i=3}^{p} (i - 2) r_i = p - 2.$$

A similar argument applied to the exterior regions produces the fact that

$$\sum_{i=3}^{p} (i - 2) \hat{r}_i = p - 2.$$

But the last two equations together imply that

$$\sum_{i=3}^{p} (i - 2)(r_i - \hat{r}_i) = 0. \quad \square$$

Example 6.5.1. Suppose that we are given the plane graph of Figure 6.5.1 with each region being indicated as r. Then there are five regions; two are triangular and three are six-sided. We will use Grinberg's theorem to prove that this graph is not hamiltonian. Note that we could simply check all ten possibilities for the regions being interior or exterior; however, the task can be handled in an easier manner.

First, suppose that the two triangular regions were in opposite regions with respect to a hamiltonian cycle C. Then in Grinberg's formula, both $r_3 = 1$ and $\hat{r}_3 = 1$ so that this term would contribute zero to the sum. But, then the only other nonzero term involves the three six-sided regions, and there is no way to obtain a contribution of zero from this term. Thus, there can be no hamiltonian cycle in which one three-sided region is interior to C and the other is exterior to C.

Next, suppose that both three-sided regions are interior to C so that $r_3 = 2$. Then, this term contributes 2 to the sum in Grinberg's formula. The only way to negate this term is in the contribution of the six–sided regions. However, it is easy to see that no placement of the six–sided regions will result in a zero sum in Grinberg's formula. A similar argument applies when $\hat{r}_3 = 2$; thus, we conclude that the graph of Figure 6.5.1 is not hamiltonian. $\square$

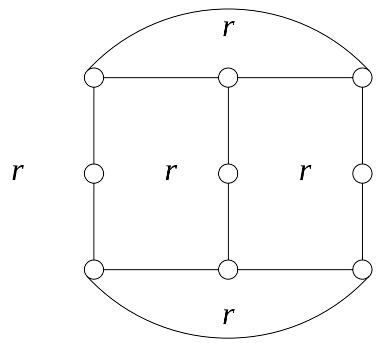


Figure 6.5.1. A plane nonhamiltonian graph.

Exercises

1. Show that if G is a plane (p, q) graph with r regions, then $p - q + r = 1 + k(G)$ where $k(G)$ is the number of components of G.

2. An $(f, d)-$ *regular polyhedron graph* is a plane graph that is d-regular $(d \geq 3)$ and each of its faces has f sides. Use Euler's formula to show that there are only five regular polyhedron graphs. (Hint: The dodecahedron is a $(5, 3)$–regular polyhedron graph.)

3. Show that "homeomorphic with" is an equivalence relation on the set of graphs.

4. Show that a graph is planar if, and only if, each of its blocks (maximal 2-connected subgraphs) is planar.

5. Let G be a maximal planar graph of order $p \geq 4$. Also let p_i denote the number of vertices of degree i in G, where $i = 3, 4, \ldots, \Delta(G) = n$. Show that
$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \cdots + (n - 6)p_n + 12.$$

6. Show that any maximal planar (p, q) graph contains a bipartite subgraph with $2\frac{q}{3}$ edges.

7. Find an example of a planar graph that contains no vertex of degree less than 5.

8. Prove that every planar graph of order $p \geq 4$ contains at least four vertices of degree at most 5.

9. Show that the Petersen graph (Figure 7.3.2) contains a subgraph homeomorphic with $K_{3,3}$ and is therefore not planar.

10. Show that if G is a connected planar (p, q) graph with girth (shortest cycle length) $g(G) = k \geq 3$, then $|E| \leq \dfrac{k(p - 2)}{(k - 2)}$.

11. Use the last result to again show that the Petersen graph is not planar.

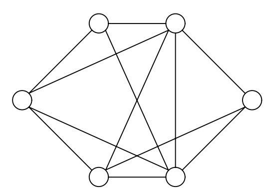
12. A graph is self-dual if it is isomorphic to its own geometric dual. Show that if G is self-dual, then $2|V| = |E| + 2$. Further, show that not every graph with this property is self-dual.

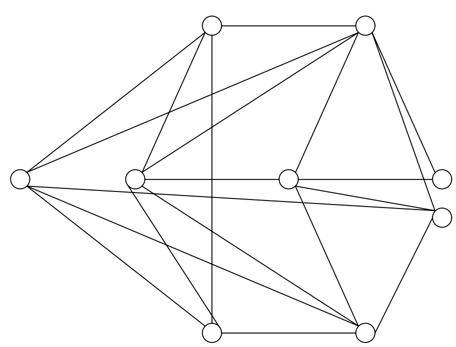
13. If G is a connected plane graph with spanning tree T and $E^* = \{ e^* \in E(G^*) \mid e \notin E(T) \}$, show that $T^* = \left(E^* \right)$ is a spanning tree of G^*.

14. Show that if $|V(G)| \geq 11$, then at least one of G and $\overline{G}$ is nonplanar.

15. Show that the average degree in a planar graph is actually less than 6. (Note that this provides an alternate proof to Corollary 6.1.2.)

16. Use the DMP algorithm to test the planarity of the following graphs.





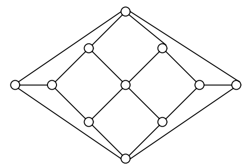
17. Prove path properties $1-5$ relating to the Hopcroft-Tarjan planarity algorithm.

18. Use the Hopcroft-Tarjan planarity algorithm to test the planarity of the graphs from exercise 16 in Chapter 6.

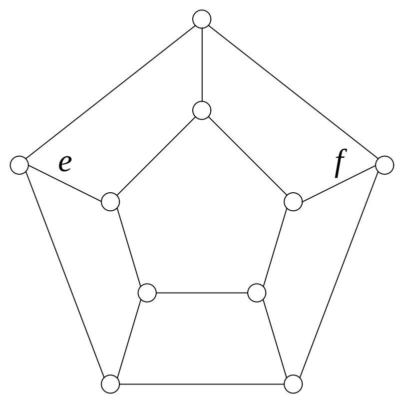
19. Prove Theorem 6.4.1.

20. How might you actually keep track of the paths both inside and outside of a given cycle? How much information must actually be recorded?

21. Use Grinberg's theorem to show that the graph below is not hamiltonian.



22. Show that no hamiltonian cycle in the graph below can contain both of the edges *e* and *f*.



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