

Answer all 6 questions. No calculators, cell-phones, class notes are allowed. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

- (15) 1(a) Translate the following argument into ***symbolic language***.
"If Adam passes, then Ben will not pass. Either Adam or Dan will pass. Therefore, if Ben passes, then Dan will pass."
- (b) Use a ***truth table*** to determine if this argument is ***logically valid***.
- (15) 2(a) Define $(\exists x \in A)[R(x)]$ and $(\forall x \in A)[S(x)]$ in terms of ***unbounded quantifiers***.
(b) Convert the formula $\neg(\exists y)(\forall z)[f(y) < g(z)} \rightarrow \{(y + z < 5) \wedge \neg(y = z)\}$ into a ***logically equivalent formula*** in which no " \neg " sign ***governs*** a ***quantifier*** or a ***connective***. [Specify which logical law you use at each step.]
- (16) 3(a) Let $\langle D_i : i \in I \rangle$ be an ***indexed family*** of sets. Define $(\bigcup_{i \in I} D_i)$ and $(\bigcap_{i \in I} D_i)$.
(b) Prove that $A - (\bigcap_{i \in I} D_i) = \bigcup_{i \in I} (A - D_i)$.
(c) Prove that $\neg(\forall x \in A)[P(x)] \Leftrightarrow (\exists x \in A)[\neg P(x)]$.
- (16) 4(a) Let R & S be relations. Define what is $S \circ R$ & define when R is a ***function***.
(b) Prove that $(R \circ S)^{-1} = (S^{-1}) \circ (R^{-1})$.
(c) Suppose F and G are functions. Prove that $(G \circ F)$ is also a function.
- (18) 5(a) Define what's an ***equivalence relation*** R on \mathbb{Z} & what's an ***equivalence class*** of R .
(b) Let R be the relation on \mathbb{Z} defined by aRb if $(b^2 - a^2)$ is an ***integer multiple*** of 12.
Prove that R is an ***equivalence relation*** on \mathbb{Z} and find the ***equivalence classes*** into which R partitions \mathbb{Z} . (Specify each equivalence class, completely.)
- (20) 6(a) Let $f: A \rightarrow B$ be a partial function from A to B . Define when f is a ***total function***, when f is ***injective***, and when f is ***surjective***?
(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the ***partial function*** defined by $f(x) = (2x-3)/(x-3)$. Prove that f is a ***total, injective, and surjective function*** from $\mathbb{R} - \{3\}$ to $\mathbb{R} - \{2\}$.
END

$$\in \forall \exists \Delta \oplus \subseteq \notin \subset \rightarrow \neg \neq \infty \emptyset \equiv \approx \leftrightarrow \times \aleph \sqrt{\nabla} \Leftrightarrow \Rightarrow \square \cong \perp \pm \geq \leq^\circ \uparrow \downarrow \perp \neg \cup \cap \mathbb{R} \mathbb{Z}$$

MAA 3200 - Introduction to Adv. Math Florida Int'l Univ.
 Solutions to Test #1 Fall 2023

1(a) Let $A = \text{Adam passes}$, $B = \text{Ben passes}$, and $D = \text{Dan passes}$. The argument says: $[(A \rightarrow \neg B) \wedge (A \vee D)] \Rightarrow (B \rightarrow D)$. Corresponding prop.

A	B	D	$(A \rightarrow \neg B) \wedge (A \vee D)$	$(B \rightarrow D)$
1	1	1	0	1
1	1	0	0	0
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	0	0
0	0	1	1	1
0	0	0	0	1

Since the corresp. proposition is a tautology, the argument is valid.

2(a) $(\exists x \in A) R(x)$ means $(\exists x)[(x \in A) \wedge R(x)]$ $(\forall x \in A) S(x)$ means $(\forall x)[(x \in A) \rightarrow S(x)]$

$$(b) \neg(\exists y)(\forall z)[\{f(y) < g(z)\} \rightarrow \{(y+z < 5) \wedge \neg(y=z)\}]$$

$$\Leftrightarrow (\forall y)\neg(\forall z)[\neg\{f(y) < g(z)\} \vee \{(y+z < 5) \wedge \neg(y=z)\}]$$

by \exists -quantifier negation law and the conditional law

$$\Leftrightarrow (\forall y)(\exists z)\neg[\neg\{f(y) < g(z)\} \vee \{(y+z < 5) \wedge \neg(y=z)\}]$$

by \forall -quantifier negation law

$$\Leftrightarrow (\forall y)(\exists z)[\neg\neg\{f(y) < g(z)\} \wedge \neg\{(y+z < 5) \wedge \neg(y=z)\}]$$

by De Morgan's law

$$\Leftrightarrow (\forall y)(\exists z)[\{f(y) < g(z)\} \wedge \{\neg(y+z < 5) \vee \neg\neg(y=z)\}]$$

by Double-negation law and De Morgan's law

$$\Leftrightarrow (\forall y)(\exists z)[\{f(y) < g(z)\} \wedge \{\neg(y+z < 5) \vee (y=z)\}] \text{ by Double Neg. law}$$

$$3(a) \bigcup_{i \in I} D_i = \{x : (\exists i \in I)(x \in A_i)\} \quad \bigcap_{i \in I} D_i = \{x : (\forall i \in I)(x \in D_i)\}$$

$$(b) x \in A - \bigcap_{i \in I} D_i \Leftrightarrow (x \in A) \wedge (x \notin \bigcap_{i \in I} D_i) \Leftrightarrow (x \in A) \wedge [\neg(\forall i \in I)(x \in D_i)]$$

$$\Leftrightarrow (x \in A) \wedge (\exists i \in I)(x \notin D_i) \Leftrightarrow (\exists i \in I)[(x \in A) \wedge (x \notin D_i)]$$

$$\Leftrightarrow (\exists i \in I)(x \in A - D_i) \Leftrightarrow x \in \bigcup_{i \in I}(A - D_i). \quad \therefore A - \bigcap_{i \in I} D_i = \bigcup_{i \in I}(A - D_i)$$

$$(c) \neg(\forall x \in A)[P(x)] \Leftrightarrow \neg(\forall x)[(x \in A) \rightarrow P(x)] \Leftrightarrow (\exists x)\neg[\neg(x \in A) \vee P(x)]$$

$$\Leftrightarrow (\exists x)[\neg\neg(x \in A) \wedge \neg P(x)] \Leftrightarrow (\exists x)[(x \in A) \wedge \neg P(x)] \Leftrightarrow (\exists x \in A)[\neg P(x)]$$

Therefore $\neg(\forall x \in A)[P(x)] \Leftrightarrow (\exists x \in A)[\neg P(x)]$.

$$4(a) S \circ R = \{(a, c) : (\exists b)[(a, b) \in R \wedge (b, c) \in S]\}.$$

R is a function if $(\forall a)(\forall c_1)(\forall c_2)[\{(a, c_1) \in R \wedge (a, c_2) \in R\} \rightarrow (c_1 = c_2)]$.

$$(b) (c, a) \in (R \circ S)^{-1} \Leftrightarrow (a, c) \in (R \circ S) \Leftrightarrow (\exists b)[(a, b) \in S \wedge (b, c) \in R]$$

$$\Leftrightarrow (\exists b)[(b, c) \in R \wedge (a, b) \in S] \Leftrightarrow (\exists b)[(c, b) \in R^{-1} \wedge (b, a) \in S^{-1}]$$

$$\Leftrightarrow (c, a) \in (S^{-1}) \circ (R^{-1}). \text{ Hence } (R \circ S)^{-1} = (S^{-1}) \circ (R^{-1}).$$

(c) Suppose $[(a, c_1) \in G \circ F] \wedge [(a, c_2) \in G \circ F]$. Then

$$(\exists b_1)[(a, b_1) \in F \wedge (b_1, c_1) \in G] \text{ and } (\exists b_2)[(a, b_2) \in F \wedge (b_2, c_2) \in G]$$

So $(a, b_1) \in F \wedge (a, b_2) \in F$. Since F is a function, $b_1 = b_2$.

$\therefore (b_1, c_1) \in G \wedge (b_1, c_2) \in G$. Since G is a function, $c_1 = c_2$.

$$\text{So } (\forall a)(\forall c_1)(\forall c_2)[\{(a, c_1) \in (G \circ F) \wedge (a, c_2) \in (G \circ F)\} \rightarrow (c_1 = c_2)]$$

Hence $G \circ F$ is a function.

5(a) A relation R on \mathbb{Z} is an equivalence relation if

$$(i) (\forall a \in \mathbb{Z})[aRa], \quad (ii) (\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa] \quad \& \quad (iii) (\forall a, b, c \in \mathbb{Z})[aRb \wedge bRc \rightarrow aRc]$$

An equivalence class of an equiv. relation R on \mathbb{Z}

is any set of the form $\{x : xRa\}$ where a is an element of \mathbb{Z} .

(b) Let $a \in \mathbb{Z}$. Then $a^2 - a^2 = 0 = 12(0)$. So $(\forall a \in \mathbb{Z})[aRa]$.

Now let $a, b \in \mathbb{Z}$ & suppose aRb . Then $b^2 - a^2 = 12k$ for some $k \in \mathbb{Z}$.

So $a^2 - b^2 = -12k = 12(-k)$ & thus bRa b.c. $-k \in \mathbb{Z}$. $\therefore (\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa]$

Finally, suppose $aRb \wedge bRc$. Then $b^2 - a^2 = 12k$ & $c^2 - b^2 = 12l$ for some $k, l \in \mathbb{Z}$. So $c^2 - a^2 = (c^2 - b^2) + (b^2 - a^2) = 12(k+l)$. $\therefore aRc$ because $k+l \in \mathbb{Z}$.

$\therefore (\forall a, b, c \in \mathbb{Z})[aRb \wedge bRc \rightarrow aRc]$. Hence R is an equivalence relation on \mathbb{Z} .

$$(c) 0^2 \equiv 0 \pmod{12}, \quad 3^2 \equiv 9 \pmod{12}, \quad 6^2 = 36 \equiv 0 \pmod{12}, \quad 9^2 = 81 \equiv 9 \pmod{12},$$

$$1^2 \equiv 1 \pmod{12}, \quad 4^2 = 16 \equiv 4 \pmod{12}, \quad 7^2 = 49 \equiv 1 \pmod{12}, \quad 10^2 = 100 \equiv 4 \pmod{12},$$

$$2^2 \equiv 4 \pmod{12}, \quad 5^2 = 25 \equiv 1 \pmod{12}, \quad 8^2 = 64 \equiv 4 \pmod{12}, \quad 11^2 = 121 \equiv 1 \pmod{12}.$$

So the equivalence classes into which R partitions \mathbb{Z} are:

$$[0]_R = [0]_{12} \cup [6]_{12} = \{12k : k \in \mathbb{Z}\} \cup \{12k+6 : k \in \mathbb{Z}\}, \quad [3]_R = [3]_{12} \cup [9]_{12} = \{12k+3 : k \in \mathbb{Z}\}$$

$$[1]_R = [1]_{12} \cup [7]_{12} \cup [5]_{12} \cup [9]_{12} = \{12k+1 : k \in \mathbb{Z}\} \cup \{12k+5 : k \in \mathbb{Z}\}$$

$$[2]_R = [2]_{12} \cup [10]_{12} \cup [4]_{12} \cup [8]_{12} = \{12k+2 : k \in \mathbb{Z}\} \cup \{12k+4 : k \in \mathbb{Z}\}$$

to(a) $f: A \rightarrow B$ is a total function if $(\forall a \in A)(\exists b \in B)[f(a) = b]$

$f: A \rightarrow B$ is injective if $(\forall a_1, a_2 \in A)[\{f(a_1) = f(a_2)\} \rightarrow (a_1 = a_2)]$

$f: A \rightarrow B$ is surjective if $(\forall b \in B)(\exists a \in A)[f(a) = b]$

(b) $f(x) = (2x-3)/(x-3) = [2(x-3)+3]/(x-3) = 2 + 3/(x-3)$.

f is a total function from $\mathbb{R} - \{3\}$ to $\mathbb{R} - \{2\}$ because $f(x)$ is defined for each $x \in \mathbb{R} - \{3\}$ (since x is never 3) and because $f(x) \in \mathbb{R} - \{2\}$, since $f(x) = 2 + 3/(x-3)$ & $3/(x-3)$ is never 0.

Suppose $f(x_1) = f(x_2)$, where $x_1, x_2 \in \mathbb{R} - \{3\}$. Then

$$2 + \frac{3}{x_1-3} = 2 + \frac{3}{x_2-3}. \quad \text{So } \frac{3}{x_1-3} = \frac{3}{x_2-3}. \quad \therefore \frac{x_1-3}{3} = \frac{x_2-3}{3}$$

Thus $x_1-3 = x_2-3$. Hence $x_1 = x_2$. So $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \therefore f$ is injective.

Let y be any element of $\mathbb{R} - \{2\}$. We will find an $x \in \mathbb{R} - \{3\}$ such that $f(x) = y$. Suppose $y = f(x) = 2 + 3/(x-3)$.

$$\text{Then } y-2 = \frac{3}{x-3}. \quad \text{So } \frac{y-2}{3} = \frac{1}{x-3} \text{ and thus } x-3 = \frac{3}{y-2}.$$

Thus $x = 3 + \frac{3}{y-2}$. We will now check that $x \in \mathbb{R} - \{3\}$ & $f(x) = y$.

$x \in \mathbb{R} - \{3\}$ because $x = 3 + \frac{3}{y-2}$ and $\frac{3}{y-2}$ is never 0.

$$\begin{aligned} \text{Also } f(x) &= 2 + \frac{3}{x-3} = 2 + \frac{3}{(3 + \frac{3}{y-2}) - 3} = 2 + \frac{3}{3/(y-2)} \\ &= 2 + \frac{3(y-2)}{3} = 2 + (y-2) = y. \end{aligned}$$

So $x \in \mathbb{R} - \{3\}$ & $f(x) = y \therefore f$ is surjective. END.