

Answer all 6 questions. No calculators, notes, or on-line stuff are allowed. An unjustified answer will receive little or no credit. Begin each of the 6 questions on 6 separate pages.

- (15) 1. (a) If R is a relation from A to B , define exactly when R is a **total function**.
If $f: A \rightarrow B$ is a total function, define exactly what is an **inverse** of f .
(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function with $f(x) = (3x-7)/(x-2)$ if $x \neq 2$; & $f(2) = 3$.
Find $f^{-1}(x)$ for each $x \in \mathbb{R}$ (*show how you got your answer*).
- (20) 2. (a) Let $g: X \rightarrow Y$ be a function and suppose that $A, B \subseteq X$ & $C, D \subseteq Y$.
Define what are the **image** $g[A]$ of A and the **pre-image** $g^{-1}[C]$ of C .
(b) Is it always true that $g[A] \cap g[B] \subseteq g[A \cap B]$?
(c) Is it always true that $g^{-1}[C] - g^{-1}[D] \subseteq g^{-1}[C - D]$?
- (15) 3. (a) Using quantifiers, write down the **First Principle of Math. Induction** for \mathbb{N} .
Also define what is an **infinite sequence** of elements from a non-empty set A .
(b) Prove that for each $n \in \mathbb{N}$, $(6^n - 5n - 1)$ is an **integer-multiple** of 25.
- (15) 4. (a) Define exactly when the set A is **finite** & when the set B is **denumerable**.
(b) Prove that $\mathbb{N} \times \mathbb{Z}^+$ is a denumerable set. [*If you claim that a function is a bijection, then you must prove that the function is indeed a bijection.*]
- (15) 5. (a) Use quantifiers to define what is a **Cauchy sequence** $\langle c_n \rangle_{n \in \mathbb{N}}$ of real numbers.
(b) If $\langle a_n \rangle_{n \in \mathbb{N}}$ and $\langle b_n \rangle_{n \in \mathbb{N}}$ are Cauchy sequences, prove that $\langle 2 \cdot a_n + 3 \cdot b_n \rangle_{n \in \mathbb{N}}$ is also a **Cauchy sequence**. (*No theorems from class or the book are allowed.*)
- (20) 6. (a) By using quantifiers, define when the sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ of reals is **convergent**.
(b) Prove that if $\langle a_n \rangle_{n \in \mathbb{N}}$ is a convergent sequence, then it is a **bounded sequence**.
(c) Suppose $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A , prove that $\langle (a_n)^2 \rangle_{n \in \mathbb{N}}$ converges to A^2 .
(*No theorems from class or the book are allowed.*)

$\in \forall \exists \Delta \oplus \subseteq \notin \subset \rightarrow \neg \neq \infty \emptyset \equiv \approx \leftrightarrow \times \mathbb{N} \sqrt{\ } \nabla \square \cong \perp \pm \geq \leq \circ \uparrow \downarrow \perp - \cup \cap \mathbb{R} \mathbb{Z} \langle \rangle \mathbb{N}$

1(a) R is a total function from A to B if R is single-valued & total:
 $(\forall a \in A)(\forall b, c \in B)[aRb \wedge aRc \rightarrow b=c] \wedge (\forall a \in A)(\exists b \in B)[aRb]$

An inverse of $f: A \rightarrow B$ is any function $g: B \rightarrow A$ such that
 $(\forall a \in A)[g(f(a)) = a] \wedge (\forall b \in B)[f(g(b)) = b]$.

(b) Let $y = f(x)$. Then $x = f^{-1}(y)$. Now $y = f(x) = (3x-7)/(x-2)$
 $= \frac{3(x-2)+6-7}{x-2} = 3 - \frac{1}{x-2}$. So $(y-3) = -1/(x-2)$ and thus
 $x-2 = -1/(y-3)$. Hence $x = 2 - \frac{1}{y-3} = f^{-1}(y)$. $\therefore f^{-1}(x) = 2 - \frac{1}{x-3}$
 when $x \neq 3$; and 2 when $x = 3$. By the way $2 - \frac{1}{x-3} = \frac{2x-7}{x-3}$
 So $f^{-1}(x) = \begin{cases} (2x-7)/(x-3) & \text{if } x \neq 3 \\ 2 & \text{if } x = 3. \end{cases}$

2(a) $g[A] = \{g(x) : x \in A\}$ and $g^{-1}[C] = \{x \in X : g(x) \in C\}$

(b) NO. Let $y \in g[A] \cap g[B]$. Then $y \in g[A]$ and $y \in g[B]$.

So $(\exists x_1 \in A)[y = g(x_1)]$ and $(\exists x_2 \in B)[y = g(x_2)]$. But

x_1 may not be equal to x_2 , so we can't say $x_1 = x_2 \in A \cap B$,
 and then conclude that $y = g(x_1) \in g[A \cap B]$.

Let $g: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $g(x) = x^2$, and put $A = \{-3, 2\}$ &

$B = \{2, 3\}$. Then $g[A] = \{4, 9\}$ and $g[B] = \{4, 9\}$ & $A \cap B = \{2\}$

So $g[A \cap B] = \{g(2)\} = \{4\}$. Hence $g[A] \cap g[B] = \{4, 9\} \not\subseteq \{4\} = g[A \cap B]$.

So in general $g[A] \cap g[B]$ is not always a subset of $g[A \cap B]$.

(c) YES. Let $x \in g^{-1}[C] - g^{-1}[D]$. Then $x \in g^{-1}[C]$ & $x \notin g^{-1}[D]$.

So $g(x) \in C$ & $g(x) \notin D$. $\therefore g(x) \in C - D$. So $x \in g^{-1}[C - D]$.

Hence $g^{-1}[C] - g^{-1}[D] \subseteq g^{-1}[C - D]$. So it is always true
 that $g^{-1}[C] - g^{-1}[D] \subseteq g^{-1}[C - D]$.

3(a) First Principle of Math Induction: Let $P(n)$ be a formula
 of First Order Logic with free variable n . Then

$\{P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]\} \Rightarrow (\forall n \in \mathbb{N})[P(n)]$. An infinite
 sequence of elements of A is just a function $s: \mathbb{N} \rightarrow A$.

3(b) Let $P(n)$ be the formula $(\exists k \in \mathbb{Z}) [6^n - 5n - 1 = 25k]$. Since $6^0 - 5(0) - 1 = 1 - 0 - 1 = 0 = 25(0)$, $P(0)$ is true. Now suppose $P(n)$ was true. Then $6^n - 5n - 1 = 25k$ for some $k \in \mathbb{Z}$. So

$$6^{n+1} - 5(n+1) - 1 = 6(6^n - 5n - 1) + 6(5n+1) - 5(n+1) - 1$$

$$= 6(25k) + 30n + 6 - 5n - 5 - 1$$

$$= 6(25k) + 25n = 25(6k + n).$$

Hence $P(n+1)$ will be true. So $P(0) \wedge (\forall n \in \mathbb{N}) [P(n) \rightarrow P(n+1)]$
 $\therefore (\forall n \in \mathbb{N}) [P(n)]$ will be true. So $6^n - 5n - 1$ will always be an integer multiple of 25 for each $n \in \mathbb{N}$.

4(a) The set A is finite if $(\exists k \in \mathbb{N}) \wedge (\exists \text{ a bijection } g: A \rightarrow \mathbb{N}_k)$

The set B is denumerable if $(\exists \text{ a bijection } f: B \rightarrow \mathbb{N})$

(b) Let $f: \mathbb{N} \times \mathbb{Z}^+ \rightarrow \mathbb{N}$ be defined by $f(k, l) = 2^k(2l-1) - 1$. Then $f(0, 1) = 2^0(2(1)-1) - 1 = 1(1) - 1 = 0 = \text{smallest element of } \mathbb{N}$.

So it appears that f might be a bijection. Let us check this.

Suppose $f(k_1, l_1) = f(k_2, l_2)$. Then $2^{k_1}(2l_1-1) - 1 = 2^{k_2}(2l_2-1) - 1$.

So $2^{k_1}(2l_1-1) = 2^{k_2}(2l_2-1)$. $\therefore 2^{k_1} = 2^{k_2}$ and so $k_1 = k_2$

(because $2l_1-1$ & $2l_2-1$ were both odd). So $2^{k_1}(2l_1-1) =$

$2^{k_1}(2l_2-1)$ [because $k_1 = k_2$]. Hence $2l_1-1 = 2l_2-1$ & so $l_1 = l_2$.

Thus $f(k_1, l_1) = f(k_2, l_2) \Rightarrow \langle k_1, l_1 \rangle = \langle k_2, l_2 \rangle$. Hence f is injective.

Now take any $n \in \mathbb{N}$ and express $n+1$ in the form $2^k(2l-1)$

Then $n+1 = 2^k(2l-1)$ and so $n = 2^k(2l-1) - 1 = f(k, l)$.

Hence f is surjective. $\therefore f$ is indeed a bijection and

so $\mathbb{N} \times \mathbb{Z}^+$ is denumerable

5(a) $\langle c_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence of real numbers if

$(\forall \epsilon > 0) (\exists N \in \mathbb{N}) (\forall m, n \geq N) [|a_m - a_n| < \epsilon]$.

(b) Let $\epsilon > 0$ be given. Then $\epsilon/5 > 0$. Since $\langle a_n \rangle$ & $\langle b_n \rangle$ are Cauchy sequences $(\exists N_1 \in \mathbb{N}) (\forall m, n \geq N_1) [|a_m - a_n| < \epsilon/5]$ and

5(b) $(\exists N_2 \in \mathbb{N})(\forall m, n \geq N_2) [|b_m - b_n| < \varepsilon/5]$. Let $N = \max\{N_1, N_2\}$.

Then $(\forall m, n \geq N)$ we have

$$\begin{aligned} |(2a_m + 3b_m) - (2a_n + 3b_n)| &= |2(a_m - a_n) + 3(b_m - b_n)| \\ &\leq |2(a_m - a_n)| + |3(b_m - b_n)| = 2|a_m - a_n| + 3|b_m - b_n| < 2 \cdot \frac{\varepsilon}{5} + 3 \cdot \frac{\varepsilon}{5} = \varepsilon \end{aligned}$$

Since ε was arbitrary, it follows that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N) [|2a_m + 3b_m - 2a_n - 3b_n| < \varepsilon]$$

So $\langle 2a_n + 3b_n \rangle_{n \in \mathbb{N}}$ is also a Cauchy sequence.

6(a) $\langle a_n \rangle_{n \in \mathbb{N}}$ is convergent if $(\exists A \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N) [|a_n - A| < \varepsilon]$.

(b) Suppose $\langle a_n \rangle$ is convergent. Then we can find an $A \in \mathbb{R}$ to which $\langle a_n \rangle$ converges. Take $\varepsilon_0 = 1$. Then $(\exists N \in \mathbb{N})(\forall n \geq N) [|a_n - A| < \varepsilon_0]$. So $(\forall n \geq N) [-1 < a_n - A < 1]$. Hence $(\forall n \geq N) [A - 1 < a_n < A + 1]$. Now let

$$L = \min \{ a_0, a_1, a_2, \dots, a_{N-1}, A - 1 \}$$

& $U = \max \{ a_0, a_1, a_2, \dots, a_{N-1}, A + 1 \}$. Then $U, L \in \mathbb{R}$ and $(\forall n \in \mathbb{N}) [L \leq a_n \leq U]$. Hence $\langle a_n \rangle$ is bounded. So if $\langle a_n \rangle$ is convergent, then it is bounded sequence.

(c) Suppose $\langle a_n \rangle$ converges to A . Then we can find an $M \in \mathbb{R}$ such that $(\forall n \in \mathbb{N}) [|a_n| < M]$. (Just take $M = \max\{|L| + 1, |U| + 1\}$ from part (b) above.)

Let $\varepsilon > 0$ be given. Put $\varepsilon' = \varepsilon / (M + |A|)$. Then $\varepsilon' > 0$.

Since $\langle a_n \rangle$ converges to A , we can find an $N \in \mathbb{N}$ such that $(\forall n \geq N) [|a_n - A| < \varepsilon']$. So for all $n \geq N$,

$$\begin{aligned} |a_n^2 - A^2| &= |(a_n + A)(a_n - A)| = |a_n + A| \cdot |a_n - A| \\ &\leq (|a_n| + |A|) \cdot |a_n - A| \\ &< (M + |A|) \cdot \varepsilon' = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arb., $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N) [|a_n^2 - A^2| < \varepsilon]$. Hence $\langle (a_n)^2 \rangle$ converges to A^2 . END.