

Answer all 6 questions. No calculators, cell-phones, or notes are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1(a) Translate the following argument into *symbolic language*.
"Either Amy or Carla will marry. If Carla marries, then Beth will not marry. Therefore, if Beth marries, then Amy will marry."
- (b) Use a *truth table* to determine if this argument is *logically valid*.
- (15) 2(a) Define $(\forall x \in A)[P(x)]$ and $(\exists x \in B)[Q(x)]$ in terms of *unbounded quantifiers*.
(b) Convert the formula $\neg(\forall y)(\exists z)[\{g(y) < h(z)\} \rightarrow \{(y+z=3) \wedge \neg(y>z)\}]$ into a *logically equivalent formula* in which no " \neg " sign governs a quantifier or a connective. [Specify which logical law you use at each step.]
- (16) 3(a) Let $\langle B_i : i \in I \rangle$ be an *indexed family* of sets. Define $\bigcap_{i \in I} (B_i)$ and $\bigcup_{i \in I} (B_i)$.
(b) Prove that $D - \bigcup_{i \in I} (B_i) = \bigcap_{i \in I} (D - B_i)$.
(c) Prove that $\neg(\exists x \in A)[Q(x)] \Leftrightarrow (\forall x \in A)[\neg Q(x)]$.
- (16) 4(a) Let R & S be relations. Define $S \circ R$ and define when R is a *function*.
(b) Let R , S , and T be any relations. Prove that $(T \circ S) \circ R = T \circ (S \circ R)$.
(c) Suppose F and G are functions. Prove that $F \circ G$ is also a function.
- (18) 5(a) Define what is an *equivalence relation* R on a set A .
(b) Let R be the relation on \mathbb{Z} defined by aRb if $(a^3 - b^3)$ is an integer multiple of 8. Prove that R is an *equivalence relation* on \mathbb{Z} and find the *equivalence classes* into which R partition \mathbb{Z} .
- (20) 6(a) Let $f: A \rightarrow B$ be a *total function*. Define when exactly is f *injective* and when exactly is f *surjective*?
(b) Let $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{2\}$ be the total function defined by $f(x) = (2x-5)/(x-3)$. Prove that $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{2\}$ is *injective & surjective*, and find $f^{-1}(x)$.

(1)

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1(a) Let $A = \text{Amy marries}$, $B = \text{Beth marries}$, & $C = \text{Cindy marries}$. The argument says $[(A \vee C) \wedge (C \rightarrow \neg B)] \Rightarrow (B \rightarrow A)$. Corresponding proposition

$$(b) \begin{array}{|c|c|c|} \hline A & B & C \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline (A \vee C) & (C \rightarrow \neg B) & [(A \vee C) \wedge (C \rightarrow \neg B)] \Rightarrow (B \rightarrow A) \\ \hline 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ \hline \end{array}$$

Since the corresponding proposition is a tautology, the argument is logically valid.

2(a) $(\forall x \in A)[P(x)]$ means $(\forall x)[(x \in A) \rightarrow P(x)]$ & $(\exists x \in B)[Q(x)]$ means $(\exists x)[(x \in B) \wedge Q(x)]$

$$(b) \neg(\forall y)(\exists z)[\{g(y) < h(z)\}] \rightarrow \{(y+z=3) \wedge \neg(y>z)\}$$

$$\Leftrightarrow (\exists y)\neg(\exists z)[\neg\{g(y) < h(z)\} \vee \{(y+z=3) \wedge \neg(y>z)\}]$$

by the \forall -quantifier negation law & the conditional law

$$\Leftrightarrow (\exists y)(\forall z)[\neg\{g(y) < h(z)\} \vee \{(y+z=3) \wedge \neg(y>z)\}]$$

by the \exists -quantifier negation law

$$\Leftrightarrow (\exists y)(\forall z)[\neg\neg\{g(y) < h(z)\} \wedge \neg\{(y+z=3) \wedge \neg(y>z)\}]$$

by de Morgan's law

$$\Leftrightarrow (\exists y)(\forall z)[\{g(y) < h(z)\} \wedge \{\neg(y+z=3) \vee \neg\neg(y>z)\}]$$

by Double negation law & deMorgan's law

$$\Leftrightarrow (\exists y)(\forall z)[\{g(y) < h(z)\} \wedge \{\neg(y+z=3) \vee (y>z)\}]$$

by double negation.

$$\Leftrightarrow (\exists y)(\forall z)[\{g(y) < h(z)\} \wedge \{(y+z=3) \rightarrow (y>z)\}] \text{ conditional law}$$

$$3(a) \bigcap_{i \in I} B_i = \{x : (\forall i \in I)(x \in B_i)\} \text{ & } \bigcup_{i \in I} B_i = \{x : (\exists i \in I)(x \in B_i)\}$$

$$(b) x \in D - (\bigcup_{i \in I} B_i) \Leftrightarrow (x \in D) \wedge \neg(x \in \bigcup_{i \in I} B_i)$$

$$\Leftrightarrow (x \in D) \wedge \neg(\exists i \in I)(x \in B_i) \Leftrightarrow (x \in D) \wedge (\forall i \in I)\neg(x \in B_i)$$

$$\Leftrightarrow (\forall i \in I)[(x \in D) \wedge \neg(x \in B_i)] \Leftrightarrow (\forall i \in I)[x \in (D - B_i)]$$

$$\Leftrightarrow x \in \bigcap_{i \in I}(D - B_i). \text{ Hence } D - (\bigcup_{i \in I} B_i) = \bigcap_{i \in I}(D - B_i).$$

(2)

$$\begin{aligned}
 3(c) \neg (\exists x \in A)[Q(x)] &\Leftrightarrow \neg (\exists x)[(x \in A) \wedge Q(x)] \\
 &\Leftrightarrow (\forall x)\{\neg [x \in A \wedge Q(x)]\} \Leftrightarrow (\forall x)[\neg(x \in A) \vee \neg Q(x)] \\
 &\Leftrightarrow (\forall x)[(x \in A) \rightarrow \neg Q(x)] \Leftrightarrow (\forall x \in A)[\neg Q(x)].
 \end{aligned}$$

4(a) $S \circ R = \{(a, c) : (\exists b)[(a, b) \in R \wedge (b, c) \in S]\}$

R is a function if $(\forall a)(\forall c_1)(\forall c_2)[\{(a, c_1) \in R \wedge (a, c_2) \in R\} \rightarrow (c_1 = c_2)]$

$$(b) (a, d) \in (T \circ S) \circ R \Leftrightarrow (\exists b)[(a, b) \in R \wedge (b, d) \in T \circ S]$$

$$\Leftrightarrow (\exists b)[(a, b) \in R \wedge (\exists c)[(b, c) \in S \wedge (c, d) \in T]]$$

$$\Leftrightarrow (\exists c)(\exists b)[\{(a, b) \in R \wedge (b, c) \in S\} \wedge (c, d) \in T]$$

$$\Leftrightarrow (\exists c)[(a, c) \in (S \circ R) \wedge (c, d) \in T] \Leftrightarrow (c, d) \in T \circ (S \circ R)$$

$$\therefore (T \circ S) \circ R = T \circ (S \circ R).$$

(c) Suppose $(a, c_1) \in F \circ G$ & $(a, c_2) \in F \circ G$. Then

$$(\exists b_1)[(a, b_1) \in G \wedge (b_1, c_1) \in F] \text{ and } (\exists b_2)[(a, b_2) \in G \wedge (b_2, c_2) \in F]$$

Since G is a function $(a, b_1) \in G \wedge (a, b_2) \in G$, implies $b_1 = b_2$

Since F is a function $(b_1, c_1) \in F \wedge (b_1, c_2) \in F$ implies $c_1 = c_2$

[Remember $b_1 = b_2$]. So $(a, c_1) \in F \circ G$ & $(a, c_2) \in F \circ G$ implies

$c_1 = c_2$. Hence $F \circ G$ is a function from the def. of a function.

5(a) A relation R on \mathbb{Z} is an equivalence relation on \mathbb{Z} if

$$(i) (\forall a \in \mathbb{Z})[aRa], (ii) (\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa], \text{ & (iii)} (\forall a, b, c \in \mathbb{Z})[(aRb \wedge bRc) \rightarrow aRc].$$

(b) Let $a \in \mathbb{Z}$. Then $a^3 - a^3 = 0 = 8(0)$. So $(\forall a \in \mathbb{Z})[aRa]$. Now let $a, b \in \mathbb{Z}$,

& suppose aRb . Then $a^2 - b^2 = 8k$ for some $k \in \mathbb{Z}$. So $b^2 - a^2 = -8k = 8(-k)$

and thus bRa . So $(\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa]$. Finally suppose $a, b, c \in \mathbb{Z}$

& suppose $aRb \wedge bRc$. Then $a^2 - b^2 = 8k$ & $b^2 - c^2 = 8l$ for some $k, l \in \mathbb{Z}$.

So $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 8k + 8l = 8(k+l)$. Thus aRc . Hence

$(\forall a, b, c \in \mathbb{Z})[(aRb \wedge bRc) \rightarrow aRc]$. So R is an equiv. relation on \mathbb{Z} .

$$(c) 0^3 \equiv 0 \pmod{8}, 2^3 \equiv 8 \equiv 0 \pmod{8}, 4^3 \equiv 64 \equiv 0 \pmod{8}, 6^3 \equiv 36 \equiv 4 \pmod{8}, 6^3 \equiv 36 \equiv 0 \pmod{8}$$

$$= (2k : k \in \mathbb{Z}), 1^3 \equiv 1 \pmod{8}, 3^3 \equiv 27 \equiv 3 \pmod{8}, 5^3 \equiv 125 \equiv 5 \pmod{8}, 7^3 \equiv 343 \equiv 7 \pmod{8}$$

$$\therefore [0]_R = [0]_8 \cup [2]_8 \cup [4]_8 \cup [6]_8 = \{8k : k \in \mathbb{Z}\} \cup \{8k+2 : k \in \mathbb{Z}\} \cup \{8k+4 : k \in \mathbb{Z}\} \cup \{8k+6 : k \in \mathbb{Z}\}$$

$$[1]_R = \{8k+1 : k \in \mathbb{Z}\}, [3]_R = \{8k+3 : k \in \mathbb{Z}\}, [5]_R = \{8k+5 : k \in \mathbb{Z}\}, [7]_R = \{8k+7 : k \in \mathbb{Z}\}$$

(3)

- 6(a) $f: A \rightarrow B$ is injective if $(\forall a_1, a_2 \in A) [f(a_1) = f(a_2)] \rightarrow (a_1 = a_2)$
 $f: A \rightarrow B$ is surjective if $(\forall b \in B) (\exists a \in A) [f(a) = b]$.

(b)(i) $f(x) = (2x-5)/(x-3) = [2(x-3)+1]/(x-3) = 2 + 1/(x-3)$.

Now suppose $x_1, x_2 \in \mathbb{R} - \{3\}$ and $f(x_1) = f(x_2)$. Then

$$2 + \frac{1}{x_1-3} = 2 + \frac{1}{x_2-3}, \text{ So } \frac{1}{x_1-3} = \frac{1}{x_2-3} \text{ & thus } x_1-3 = x_2-3.$$

Hence $x_1 = x_2$. So $f(x_1) = f(x_2) \Rightarrow (x_1 = x_2)$. $\therefore f$ is injective

(ii) Let y be any element of $\mathbb{R} - \{2\}$. We will find an $x \in \mathbb{R} - \{3\}$ such that $f(x) = y$. Now suppose $y = f(x) = 2 + \frac{1}{x-3}$.

$$\text{Then } y-2 = \frac{1}{x-3} \text{ & so } \frac{1}{y-2} = x-3. \text{ Thus } x = 3 + \frac{1}{y-2}.$$

We now have to check that $x \in \mathbb{R} - \{3\}$ & $f(x) = y$.

First of all, x is never 3 because $x = 3 + \frac{1}{y-2}$ & $1/(y-2)$ is never zero. So $x \in \mathbb{R} - \{3\}$. Also

$$f(x) = 2 + \frac{1}{x-3} = 2 + \frac{1}{(3 + \frac{1}{y-2})-3} = 2 + \frac{1}{1/(y-2)}$$

Hence $f(x) = y$. $\therefore f$ is surjective

(iii) Finally let $y = f^{-1}(x)$. [$f^{-1}(x)$ exists because f is a bijection]

$$\text{Then } f(y) = f(f^{-1}(x)) = x, \text{ So } 2 + \frac{1}{y-3} = x$$

$$\therefore \frac{1}{y-3} = x-2. \text{ So } y-3 = \frac{1}{x-2} \text{ and hence } y = 3 + \frac{1}{x-2}$$

$$\text{But } y = f^{-1}(x). \text{ So } f^{-1}(x) = 3 + \frac{1}{x-2} = \frac{3(x-2)+1}{x-2} = \frac{3x-5}{x-2}.$$

Note $f^{-1}: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$.

END