

Answer all 6 questions. **No calculators, notes, or on-line data are allowed.** An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1(a) Translate the following argument into **symbolic language**.
"Either Adam or Ben will win gold. If Ben wins gold, then Carl will not win gold. Therefore, if Carl wins gold, then Adam will win gold."
- (b) Use a **truth table** to determine if this argument is **logically valid**.
- (15) 2(a) Define $(\forall x \in A)[R(x)]$ and $(\exists x \in B)[S(x)]$ in terms of *unbounded quantifiers*.
(b) Convert the formula $\neg(\exists y)(\forall z)[\{f(y) > g(z)\} \rightarrow \{(y + z > 5) \wedge \neg(y = z)\}]$ into a *logically equivalent formula* in which no " \neg " sign **governs** a quantifier or a connective. [Specify which law you use at each step.]
- (15) 3(a) Let $\langle A_i : i \in I \rangle$ be an *indexed family* of subsets of a universal set U . Define what are $\bigcup_{i \in I} (A_i)$ and $\bigcap_{i \in I} (A_i)$. Let B also be a subset of U .
(b) Prove that $B - \bigcup_{i \in I} (A_i) = \bigcap_{i \in I} (B - A_i)$.
- (15) 4(a) Define what is a *relation R*. If R & S are relations define what are R^{-1} & $S \circ R$.
(b) Let R , S , and T be any relations. Prove that $T \circ (S \circ R) = (T \circ S) \circ R$.
- (20) 5(a) Define what is an *equivalence relation R* on a set A .
(b) Let R be the relation on \mathbb{Z} defined by aRb if $(a^3 - b^3)$ is an integer multiple of 12. Prove that R is an *equivalence relation* and find the *equivalence classes* into which R partitions \mathbb{Z} . (Specify each equivalence class, completely.)
- (20) 6(a) Define what it means for the *partial function* $f: A \rightarrow B$ to be a *total function*. Also define when exactly is f *injective* and when exactly is f *surjective*?
(b) Let $f: \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{3\}$ be the partial function defined by $f(x) = (3x+2)/(x-1)$. Prove that $f: \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{3\}$ is a *total, injective, & surjective* function. END

Solutions to Test #1

SPRING 2023.

1(a) Let $A = \text{Adam wins gold}$, $B = \text{Ben wins gold}$, & $C = \text{Carl wins gold}$.

The argument says: $[(A \vee B) \wedge (B \rightarrow C)] \Rightarrow (C \rightarrow A)$, corresp. proposition

$$(b) \begin{array}{ccccccc} A & B & C & [(A \vee B) \wedge (B \rightarrow C)] & \Rightarrow & (C \rightarrow A) \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

A	B	C	$(A \vee B)$	\wedge	$(B \rightarrow C)$	\Rightarrow	$(C \rightarrow A)$
1	1	1	1	0	0	1	1
1	1	0	1	1	1	1	1
1	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1
0	1	1	1	0	0	0	0
0	1	0	1	1	1	1	1
0	0	1	0	0	1	0	0
0	0	0	0	0	1	1	1

Since the corresp. proposition is a tautology, the argument is valid.

$$2(a) (\forall x \in A)[R(x)] \text{ means } (\forall x)[(x \in A) \rightarrow R(x)]$$

$$(\exists x \in B)[S(x)] \text{ means } (\exists x)[(x \in B) \wedge S(x)]$$

$$(b) \neg(\exists y)(\forall z)[\{f(y) > f(z)\} \rightarrow \{(y+z > 5) \wedge \neg(y=z)\}]$$

$$\Leftrightarrow (\forall y)(\forall z)[\neg\{f(y) > f(z)\} \vee \{(y+z > 5) \wedge \neg(y=z)\}]$$

by \exists -quantifier negation law & the conditional law

$$\Leftrightarrow (\forall y)(\exists z)[\neg\{f(y) > f(z)\} \vee \{(y+z > 5) \wedge \neg(y=z)\}]$$

by the \forall -quantifier negation law

$$\Leftrightarrow (\forall y)(\exists z)[\neg\neg\{f(y) > f(z)\} \wedge \neg\{(y+z > 5) \wedge \neg(y=z)\}]$$

by De Morgan's law

$$\Leftrightarrow (\forall y)(\exists z)[\{f(y) > f(z)\} \wedge \{\neg(y+z > 5) \vee \neg\neg(y=z)\}]$$

by Double-negation law & De Morgan's law

$$\Leftrightarrow (\forall y)(\exists z)[\{f(y) > f(z)\} \wedge \{\neg(y+z > 5) \vee (y=z)\}]$$

by double negation law

$$3(a) \bigcup_{i \in I} (A_i) = \{x : (\exists i \in I)(x \in A_i)\} \quad \bigcap_{i \in I} (A_i) = \{x : (\forall i \in I)(x \in A_i)\}$$

$$(b) x \in B - \bigcup_{i \in I} A_i \Leftrightarrow (x \in B) \wedge [x \notin \bigcup_{i \in I} (A_i)] \Leftrightarrow (x \in B) \wedge [\neg(\exists i \in I)(x \in A_i)]$$

$$\Leftrightarrow (x \in B) \wedge (\forall i \in I)[\neg(x \in A_i)] \Leftrightarrow (\forall i \in I)[(x \in B) \wedge \neg(x \in A_i)]$$

$$\Leftrightarrow (\forall i \in I)[x \in (B - A_i)] \Leftrightarrow x \in \bigcap_{i \in I} (B - A_i).$$

$$\text{Therefore } B - \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (B - A_i).$$

4(a) A relation R is a set of only ordered pairs. $R' = \{(b, a) : (a, b) \in R\}$

$$S \circ R = \{(a, c) : (\exists b)[(a, b) \in R \wedge (b, c) \in S]\}$$

(b) First of all, $T_0(S \circ R)$ & $(T_0 S) \circ R$ are sets of ordered pairs only

$$\text{Now } (a, d) \in T_0(S \circ R) \Leftrightarrow (\exists c)\{(a, c) \in S \circ R \wedge (c, d) \in T\}$$

$$\Leftrightarrow (\exists c)\{(\exists b)[(a, b) \in R \wedge (b, c) \in S] \wedge (c, d) \in T\}$$

$$\Leftrightarrow (\exists c)(\exists b)\{[(a, b) \in R \wedge (b, c) \in S] \wedge (c, d) \in T\}$$

$$\Leftrightarrow (\exists b)\{(a, b) \in R \wedge (\exists c)[(b, c) \in S \wedge (c, d) \in T]\}$$

$$\Leftrightarrow (\exists b)\{(a, b) \in R \wedge (b, d) \in T_0 S\} \Leftrightarrow (a, d) \in (T_0 S) \circ R$$

$$\therefore T_0(S \circ R) = (T_0 S) \circ R.$$

5(a) A relation R on the set A is an equivalence relation if

$$(i) (\forall a \in A)[aRa], (ii) (\forall a, b \in A)[aRb \rightarrow bRa], \& (iii) (\forall a, b, c \in A)[(aRb \wedge bRc) \rightarrow aRc]$$

R is reflexive on A

R is symmetric

R is transitive

(b) Let $a \in \mathbb{Z}$. Then $a^3 - a^3 = 0 = 12(0)$. So $(\forall a \in \mathbb{Z})[aRa]$

Now let $a, b \in \mathbb{Z}$ and suppose aRb . Then $a^3 - b^3 = 12k$ for some $k \in \mathbb{Z}$, so $b^3 - a^3 = -(12k) = 12(-k)$. So bRa b.c. $-k \in \mathbb{Z}$. $\therefore (\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa]$.

Finally, let $a, b, c \in \mathbb{Z}$ and suppose aRb and bRc . Then

$a^3 - b^3 = 12k$ & $b^3 - c^3 = 12l$ for some $k, l \in \mathbb{Z}$. Hence

$$a^3 - b^3 = (a^3 - b^3) + (b^3 - c^3) = (12k) + (12l) = 12(k+l)$$

So $(\forall a, b, c \in \mathbb{Z})[(aRb \wedge bRc) \rightarrow aRc]$. $\therefore R$ is an equiv. relation.

$$(c) 0^3 \equiv 0 \pmod{12}, \quad 4^3 \equiv 64 \equiv 4 \pmod{12} \quad 8^3 \equiv (64)(8) \equiv 4(8) \equiv 8$$

$$1^3 \equiv 1 \pmod{12}, \quad 5^3 \equiv 25(5) \equiv (1)(5) \equiv 5 \quad 9^3 \equiv (81)(9) \equiv 81 \equiv 9$$

$$2^3 \equiv 8 \equiv 8 \pmod{12}, \quad 6^3 \equiv 36(6) \equiv 0 \pmod{12} \quad 10^3 \equiv (100)(10) \equiv 4(10) \equiv 4$$

$$3^3 \equiv 27 \equiv 3 \pmod{12}, \quad 7^3 \equiv 49(7) \equiv (1)(7) \equiv 7 \quad 11^3 \equiv (121)(11) \equiv (1)(11) \equiv 11$$

So the equivalence classes into which R partitions \mathbb{Z} are

$$[0]_R = [0]_{12} \cup [6]_{12} = \{12k : k \in \mathbb{Z}\} \cup \{12k+6 : k \in \mathbb{Z}\} = \{6k : k \in \mathbb{Z}\}$$

$$[1]_R = [1]_{12} = \{12k+1 : k \in \mathbb{Z}\} \quad [9]_R = [9]_{12} = \{12k+9 : k \in \mathbb{Z}\}$$

$$[3]_R = [3]_{12} = \{12k+3 : k \in \mathbb{Z}\} \quad [11]_R = [11]_{12} = \{12k+11 : k \in \mathbb{Z}\}$$

$$[5]_R = [5]_{12} = \{12k+5 : k \in \mathbb{Z}\} \quad [2]_R = [2]_{12} \cup [8]_{12} = \{12k+2 : k \in \mathbb{Z}\} \cup \{12k+8 : k \in \mathbb{Z}\}$$

$$[7]_R = [7]_{12} = \{12k+7 : k \in \mathbb{Z}\} \quad [4]_R = [4]_{12} \cup [10]_R = \{12k+4 : k \in \mathbb{Z}\} \cup \{12k+10 : k \in \mathbb{Z}\}$$

6(a) $f: A \rightarrow B$ is a total function if $(\forall a \in A)(\exists b \in B) [f(a) = b]$

$f: A \rightarrow B$ is injective if $(\forall a_1, a_2 \in A) [\{f(a_1) = f(a_2)\} \rightarrow (a_1 = a_2)]$

$f: A \rightarrow B$ is surjective if $(\forall b \in B)(\exists a \in A) [f(a) = b]$.

(b) $f: \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{3\}$ and $f(x) = (3x+2)/(x-1) = \frac{3(x-3)+5}{x-1} = 3 + \frac{5}{x-1}$.

f is total because for all $x \in \mathbb{R} - \{1\}$, $f(x) = 3 + 5/(x-1)$ is defined (since $x \neq 1$) & this $f(x) \in \mathbb{R} - \{3\}$ (bec. $\frac{5}{x-1}$ is never 0 so $f(x)$ can never be 3).

Suppose $f(x_1) = f(x_2)$. Then $3 + \frac{5}{x_1-1} = 3 + \frac{5}{x_2-1}$.

So $\frac{5}{x_1-1} = \frac{5}{x_2-1}$ and hence $\frac{x_1-1}{5} = \frac{x_2-1}{5}$. Thus $x_1-1 = x_2-1$

and hence $x_1 = x_2$. $\therefore f(x_1) = f(x_2) \Rightarrow (x_1 = x_2)$. Hence

$f: \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{3\}$ is injective.

Let $y \in \mathbb{R} - \{3\}$. We will find an $x \in \mathbb{R} - \{1\}$ such that $f(x) = y$.

Suppose $y = f(x) = 3 + \frac{5}{x-1}$. Then $y-3 = \frac{5}{x-1}$,

So $\frac{y-3}{5} = \frac{1}{x-1}$ and thus $x-1 = \frac{5}{y-3}$. $\therefore x = 1 + \frac{5}{y-3} \in \mathbb{R} - \{1\}$ because $5/(y-3)$ is never 0.

Let us now check that this x works, i.e., $f(x) = y$.

$$\begin{aligned} f(x) &= 3 + \frac{5}{x-1} = 3 + \frac{5}{1 + [5/(y-3)] - 1} = 3 + \frac{5}{5/(y-3)} \\ &= 3 + 5 \cdot \frac{(y-3)}{5} = 3 + (y-3) = y. \end{aligned}$$

Hence $f: \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{3\}$ is surjective. END.