

Answer all 6 questions. No calculators, notes, or on-line stuff are allowed. An unjustified answer will receive little or no credit. Draw a line to separate each of your 6 solutions to the 6 questions.

- (15) 1. (a) Let  $f: A \rightarrow B$  be a function. Define exactly what is an *inverse* of the function  $f$ .  
(b) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function with  $f(x) = (3x+2)/(x-4)$  if  $x \neq 4$ ; &  $f(4) = 3$ . Find  $f^{-1}(x)$  for each  $x \in \mathbb{R}$  (show how you got your answer).
- (20) 2. (a) Let  $g: X \rightarrow Y$  be a function and suppose that  $A, B \subseteq X$  and  $C, D \subseteq Y$ . Define what is  $g[A]$  and define what is  $g^{-1}[C]$ .  
(b) Is it always true that  $g[A] - g[B] \subseteq g[A - B]$ ?  
(c) Is it always true that  $g^{-1}[C] \cap g^{-1}[D] \subseteq g^{-1}[C \cap D]$ ?
- (15) 3. (a) Using quantifiers, write down the *First Principle of Math. Induction* for  $\mathbb{N}$ . Also define what is a *finite sequence* of elements from a non-empty set  $S$ .  
(b) Prove that  $(\forall n \in \mathbb{N}) [4^{n+2} + 5^{2n+1}$  is an *integer-multiple* of 21].
- (15) 4. (a) Define what it means for a set  $A$  to be *finite* & for a set  $B$  to be *uncountable*.  
(b) Prove that  $\mathbb{Z}^+ \times \mathbb{N} \approx \mathbb{Z}^+$ . [“ $\approx$ ” means *equivalent*. If you claim that a function is a *bijection*, then you must prove that the function is indeed a *bijection*.]
- (20) 5. (a) By using quantifiers, define what is a *convergent sequence*  $\langle a_n \rangle_{n \in \mathbb{N}}$  of real numbers, and what it means for the *sequence*  $\langle c_n \rangle_{n \in \mathbb{N}}$  to be *bounded*.  
(b) Suppose  $\langle a_n \rangle_{n \in \mathbb{N}}$  converges to  $A$ , and  $\langle b_n \rangle_{n \in \mathbb{N}}$  converges to  $B$ . Prove that  $\langle 2a_n + 7b_n \rangle_{n \in \mathbb{N}}$  is convergent. (No theorems from class are allowed.)
- (15) 6. (a) Use quantifiers to define what is a *Cauchy sequence*  $\langle a_n \rangle_{n \in \mathbb{N}}$  of real numbers  
(b) If  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a *Cauchy sequence*, prove that  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a *bounded sequence*. Also give an example to show the converse is false.

$\in \forall \exists \Delta \oplus \subseteq \notin \subset \rightarrow \neg \neq \infty \emptyset \equiv \approx \leftrightarrow \times \div \sqrt{\square} \cong \perp \pm \geq \leq \uparrow \downarrow \cup \cap \mathbb{R} \mathbb{Z} \diamond \mathbb{N}$

1 (a) An inverse of the function  $f: A \rightarrow B$  is any function  $g: B \rightarrow A$  such that  $(\forall a \in A)[g(f(a)) = a]$  and  $(\forall b \in B)[f(g(b)) = b]$ .

(b) Let  $y = f(x)$  &  $x \neq 4$ . Then  $x = f^{-1}(y)$ . Now  $y = f(x) = (3x+2)/(x-4)$

So  $y(x-4) = 3x+2$  and thus  $xy - 4y = 3x+2$ . So  $xy - 3x = 4y+2$ .

$\therefore x(y-3) = 4y+2$  and hence  $f^{-1}(y) = x = (4y+2)/(y-3)$ . Thus

$$f^{-1}(x) = (4x+2)/(x-3), x \neq 3. \therefore f^{-1}(x) = \begin{cases} (4x+2)/(x-3) & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$$

2 (a)  $g[A] = \{g(x) : x \in A\}$ ,  $g^{-1}[C] = \{x \in X : g(x) \in C\}$ .

(b) YES. Let  $y \in g[A] - g[B]$ . Then  $y \in g[A]$  and  $y \notin g[B]$ .

So we can find at least one  $x_0$  in  $A$  such that  $y = g(x_0)$ .

Now since  $y \notin g[B]$ , we can never have any  $x$  in  $B$  with  $y = g(x)$ , otherwise we would get  $y = g(x) \in g[B]$ .

So  $x_0 \in A - B$ . Since  $y = g(x_0)$  &  $x_0 \in A - B$ ,  $y \in g[A - B]$ .

Hence  $g[A] - g[B] \subseteq g[A - B]$ .

(c) YES. Let  $x \in g^{-1}[C] \cap g^{-1}[D]$ . Then  $x \in g^{-1}[C]$  &  $x \in g^{-1}[D]$ .

So  $g(x) \in C$  and  $g(x) \in D$  by the definition. Hence

$g(x) \in (C \cap D)$ . So  $x \in g^{-1}[C \cap D]$ .  $\therefore g^{-1}[C] \cap g^{-1}[D] \subseteq g^{-1}[C \cap D]$ .

3 (a) Let  $P(n)$  be a first-order formula with free variable  $n$ . Then  $\{P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]\} \Rightarrow (\forall n \in \mathbb{N})[P(n)]$ . A finite sequence of elements of  $S$  is just a function  $f: \mathbb{N}_k \rightarrow S$  for some  $k \in \mathbb{N}$ .

(b) Let  $P(n)$  be the formula:  $(\exists k \in \mathbb{Z})[4^{n+2} + 5^{2n+1} = 21k]$ . Since  $4^{0+2} + 5^{2(0)+1} = 21$ ,  $P(0)$  is true. Now suppose  $P(n)$  is true. Then  $4^{n+2} + 5^{2n+1} = 21k$  for some  $k \in \mathbb{Z}$ . So  $4^{(n+1)+2} + 5^{2(n+1)+1} = 4 \cdot (4^{n+2} + 5^{2n+1}) - 4 \cdot 5^{2n+1} + 25 \cdot 5^{2n+1}$

$= 4k + 21 \cdot 5^{2n+1} = 21(4k + 5^{2n+1})$ . So  $(\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]$ . Hence

$\{P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]\}$ . Thus  $(\forall n \in \mathbb{N})[P(n)]$ , i.e.,  $4^{n+2} + 5^{2n+1}$  is an integer multiple of 21 for each  $n \in \mathbb{N}$ .

4(a) A set  $A$  is finite if we can find a bijection  $f: A \rightarrow \mathbb{N}_k$  for some  $k \in \mathbb{N}$ .  
 A set  $B$  is uncountable if there is no bijection  $f: B \rightarrow \mathbb{N}$  and if for each  $k \in \mathbb{N}$ , there is no bijection from  $B$  to  $\mathbb{N}_k$ .

(b) Let  $f: \mathbb{Z}^+ \times \mathbb{N} \rightarrow \mathbb{Z}^+$  be defined by  $f(k, l) = 2^{k-1}(2l+1)$ .

Suppose  $f(k_1, l_1) = f(k_2, l_2)$ . Then  $2^{k_1-1}(2l_1+1) = 2^{k_2-1}(2l_2+1)$ .

Since  $2l_1+1$  &  $2l_2+1$  are odd, we must have  $2^{k_1-1} = 2^{k_2-1}$ . So  $k_1 = k_2$ .

Thus  $2l_1+1 = 2l_2+1$  & so  $l_1 = l_2$ .  $\therefore (k_1, l_1) = (k_2, l_2)$  & so  $f$  is injective.

Now let  $n \in \mathbb{Z}^+$ . Then we can write  $n$  in the form  $2^a(2b+1)$ ,

where  $a, b \in \mathbb{N}$ . So  $n = 2^{k-1}(2l+1)$  where  $k = a+1 \in \mathbb{Z}^+$  &  $l = b \in \mathbb{N}$ .

Hence  $f(k, l) = n$ . So  $f$  is surjective.  $\therefore \mathbb{Z}^+ \times \mathbb{N} \approx \mathbb{Z}^+$ .

5(a)  $\langle a_n \rangle$  is convergent if  $(\exists A \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[|a_n - A| < \varepsilon]$

$\langle c_n \rangle$  is bounded if  $(\exists L \in \mathbb{R})(\exists U \in \mathbb{R})(\forall n \in \mathbb{N})[L \leq c_n \leq U]$

(b) Fix  $\varepsilon > 0$ . Then  $\varepsilon/9 > 0$ . Since  $\langle a_n \rangle$  converges to  $A$  &  $\langle b_n \rangle$

converges to  $B$  we can find  $N_1 \in \mathbb{N}$  &  $N_2 \in \mathbb{N}$  such that

$(\forall n \geq N_1)(|a_n - A| < \varepsilon/9)$  and  $(\forall n \geq N_2)(|b_n - B| < \varepsilon/9)$ . Let  $N =$

$\max\{N_1, N_2\}$ . Then  $(\forall n \geq N)$  we have

$$\begin{aligned} |(2a_n + 7b_n) - (2A + 7B)| &= |2(a_n - A) + 7(b_n - B)| \leq |2(a_n - A)| + |7(b_n - B)| \\ &= 2|a_n - A| + 7|b_n - B| < 2(\varepsilon/9) + 7(\varepsilon/9) = \varepsilon. \end{aligned}$$

$\therefore (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|(2a_n + 7b_n) - (2A + 7B)| < \varepsilon]$ . Hence

$\langle 2a_n + 7b_n \rangle$  is convergent.

6 (a)  $\langle a_n \rangle$  is a Cauchy seq. if  $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)[|a_m - a_n| < \varepsilon]$ .

(b) Suppose  $\langle a_n \rangle$  is a Cauchy seq. Take  $\varepsilon = 1$ . Then

$(\exists N \in \mathbb{N})(\forall m, n \geq N)[|a_m - a_n| < 1]$ . In particular  $(\forall m \geq N)$

$|a_m - a_N| < 1$ , i.e.,  $(a_N - 1) < a_m < (a_N + 1)$ .

Let  $L = \min\{a_0, a_1, a_2, \dots, a_{N-1}, a_N - 1\}$  and

$U = \max\{a_0, a_1, a_2, \dots, a_{N-1}, a_N + 1\}$ . Then for all  $m \in \mathbb{N}$ ,

we have  $L \leq a_m \leq U$ . Hence  $\langle a_n \rangle$  is a bounded sequence

(c). Let  $\langle a_n \rangle = (-1)^n$ . Then  $-1 \leq a_n \leq 1$  for all  $n \in \mathbb{N}$ , but

$\langle a_n \rangle$  is not a Cauchy sequence. [Take  $\varepsilon = 1$  and  $(\forall N \in \mathbb{N})$

choose  $m = N$  &  $n = N+1$ . Then  $|a_m - a_n| = |(-1)^N - (-1)^{N+1}| = 2 \geq \varepsilon = 1$ .

So  $(\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists m, n \geq N)[|a_m - a_n| \geq \varepsilon]$ .